Transformations and Geometry

In CG objects are usually represented as pointsets $O \subseteq V$ in real valued vector spaces V such as $V = \mathbb{R}^3$ or $V = \mathbb{R}^2$ depending on the application.

One can manipulate (e.g. move, scale, rotate) these objects by applying transformations $T: V \to V$ to the corresponding point sets $O \mapsto T(O) = \{T(v) \mid \forall v \in O\}$. In CG usually only those transformations are considered, which can be handled efficiently.

▶ Linear Transformations

Definition:

A Linear transformation $f: V \to V$ satisfies

I) $f(u+v) = f(u) + f(v) \qquad \forall u, v \in V$

II) $f(cv) = c f(v) \qquad \forall v \in V \quad c \in \mathbb{R}$

Note:

Each linear transformation can be represented by a matrix M_f, $f(v) = M_f \cdot v \quad \forall v \in V$.

Example:

$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$ rotation by angle ϑ (counterclockwise)

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection against x-axis

$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ scaling by α in x-direction and β in y-direction

$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ horizontal shearing

$\begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}$ squeezing (volume preserving)

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▶ Composition of Linear Transformations

Lemma:

Let $f: V \to V$ and $g: V \to V$ be linear then the composed map $f \circ g: V \to V$ is linear. Moreover, the corresponding matrices satisfy $M_{f\circ g} = M_f M_g$.

Example:

A squeezing followed by a counterclockwise rotation with rotation angle ϑ

operation: $\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha\cos\vartheta & -\frac{1}{\alpha}\sin\vartheta \\ \alpha\sin\vartheta & \frac{1}{\alpha}\cos\vartheta \end{pmatrix}$

A rotation followed by a squeezing:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} = \begin{pmatrix} \alpha\cos\vartheta & -\alpha\sin\vartheta \\ \frac{1}{\alpha}\sin\vartheta & \frac{1}{\alpha}\cos\vartheta \end{pmatrix}$$

In general these transformations do not commute!

Special classes of transformations

- Orthogonal transformations $f: V \to V$ satisfy $M_f^T M_f = M_f M_f^T = \mathbb{1}$ and preserve length and angles.
- Volume preserving transformations $f: V \to V$ satisfy $|\det M_f| = 1$ (squeezing).
- Invertible transformations $f: V \to V$ are those for which an inverse $g: V \to V$ with $f\circ g = g\circ f = id$ exists $\Leftrightarrow \det M_f \neq 0$.
- A projection is a linear transformation $f: V \to V$ which is idempotent $f\circ f = f$.

▶ Decomposition of Linear Transformations

Is there a way to decompose a composition of multiple linear transformations? In general this is not possible. Only under very special conditions compositions can be undone.

II • Let A be a symmetric matrix ($A = A^T$), then A can always be decomposed into

$$A = RSR^T \quad \text{(eigenvalues, eigenvectors)}$$

where R is orthogonal (rotation) and S is diagonal (scaling)

- For non-symmetric matrices the singular value decomposition

$$A = USV^T$$

can be used. Here U and V are orthogonal and S is a diagonal matrix with non-negative-values.

- 2D rotations can be decomposed into shearings, which is especially usefull for the rotation of raster images (shearing leves no holes and is very efficient.

shearings

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} = \begin{pmatrix} 1 & \frac{\cos\vartheta - 1}{\sin\vartheta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin\vartheta & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\cos\vartheta - 1}{\sin\vartheta} \\ 0 & 1 \end{pmatrix}$$

Consider a raster image with raster positions $\binom{i}{j} \in \mathbb{N}^2$

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i + sj \\ j \end{pmatrix}.$$ round to nearest integer

- There are more...

► Transformation of Normal Vectors

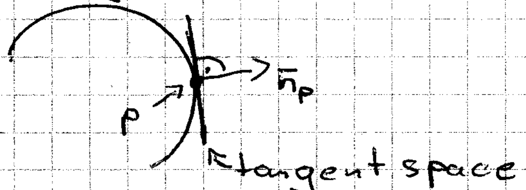
Till now we did transform positions. How do directions (normal vectors) transform?

Definition:

Consider a surface at position $p \in V$. A vector $n_p \in V$ is called normal vector if

$$n_p^T t_p = 0$$

for all tangent vectors t_p.



Lemma:
Let M_f be the transformation matrix associated to the invertible linear transformation $f: V \to V$ (e.g. used to transform an object) $\bar{n}$ a normal vector orthogonal to all $\bar{t}$ in the tangent space, then $(M_f^{-1})^T \bar{n}$ is the normal vector of the transformed tangent space with tangent vectors $M_f t$.

Proof:

$$[(M_f^{-1})^T \bar{n}]^T [M_f t] = n^T [(M_f^{-1})^T]^T M_f t$$
$$= n^T M_f^{-1} M_f t$$
$$= n^T t = 0 \qquad \square$$

Note:
- The length of the normal vector might chang under such a transformation.
- For orthogonal transformations (e.g rotations and reflection) $M^{-1} = M^T$. Hence normal vectors transform just like points in this case.

▶ Affine Transformations & Homogeneous coordinates

An important transformation, which does not fall under linear transformations is the translation

$$v \mapsto v+t, \qquad t \text{ translation vector}$$

which together with linear transformations forms the set of the so called affine transformations

$$v \mapsto Mv + t.$$

Note:
- Affine transformations map lines to lines
- —— " —— parallel lines to parallel lines
- Intersection points of lines are maped to the intersection points of the transformed lines

IV

- In general angles and length are not preserved. This holds true only for a subset of the affine transformations refered to as rigid transformations.

In order to be able to use the highly efficient vector matrix mashinery for all affine transformations one can use homogeneous coordinates. The concept will be explained at the example of the vector space $\mathbb{R}^3$, though it can be applied to all finite dimensional vector spaces.

Definition:
Let $v=(v_x, v_y, v_z) \in \mathbb{R}^3$ be a point and $n=(n_x, n_y, n_z)$ be a direction (e.g. a normal vector). In homogeneous coordinates these vectors are $(v_x, v_y, v_z, 1) \in \mathbb{R}^4$ and $(n_x, n_y, n_z, 0) \in \mathbb{R}^4$.

Lemma:
Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with associated matrix M_f. A point $v=(v_x, v_y, v_z)$ can be transformed using homogeneous coordinates by

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \mapsto \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} M_f & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\,0\,0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix} = \begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} v_x' \\ v_y' \\ v_z' \end{pmatrix} = M_f v$$

A direction (n_x, n_y, n_z) by

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \mapsto \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} (M_f^{-1})^T & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\,0\,0 & 1 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix} = \begin{pmatrix} n_x' \\ n_y' \\ n_z' \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} n_x' \\ n_y' \\ n_z' \end{pmatrix} = (M_f^{-1})^T n.$$

Let $t=(t_x, t_y, t_z) \in \mathbb{R}^3$ be a translation vector then

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \mapsto \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{1} & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ 0\,0\,0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix} = \begin{pmatrix} v_x' \\ v_y' \\ v_z' \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} v_x' \\ v_y' \\ v_z' \end{pmatrix} = v + t$$

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \mapsto \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{1} & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ 0\,0\,0 & 1 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix} = \begin{pmatrix} n_x' \\ n_y' \\ n_z' \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} n_x' \\ n_y' \\ n_z' \end{pmatrix} = n.$$

Note:

- All affine transformations can be represented as 4x4 matrix to transform points and directions represented as homogeneous coordinates.
- In their 4x4 matrix representation affine transformations can be composed by matrix multiplication.

$$\begin{pmatrix} \mathbb{1} & \bar{t} \\ \bar{0}^T & 1 \end{pmatrix}\begin{pmatrix} M_f & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix} = \begin{pmatrix} M_f & \bar{t} \\ \bar{0}^T & 1 \end{pmatrix}$$

Linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^3$

translation by $\bar{t} \in \mathbb{R}^3$

$$\begin{pmatrix} M_f & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix}\begin{pmatrix} \mathbb{1} & \bar{t} \\ \bar{0}^T & 1 \end{pmatrix} = \begin{pmatrix} M_f & M_f\bar{t} \\ \bar{0}^T & 1 \end{pmatrix}$$

- Inverse transformations are in a one-to-one correspondence to the inverse of the 4x4 matrices

$$\begin{pmatrix} \mathbb{1} & \bar{t} \\ \bar{0}^T & 1 \end{pmatrix}\begin{pmatrix} \mathbb{1} & -\bar{t} \\ \bar{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & -\bar{t} \\ \bar{0}^T & 1 \end{pmatrix}\begin{pmatrix} \mathbb{1} & \bar{t} \\ \bar{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix}$$

translation by $-\bar{t}$

translation by $\bar{t}$

$$\begin{pmatrix} M_f & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix}\begin{pmatrix} M_f^{-1} & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix} = \begin{pmatrix} M_f^{-1} & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix}\begin{pmatrix} M_f & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \bar{0} \\ \bar{0}^T & 1 \end{pmatrix}$$

Linear transformation $f^{-1}: \mathbb{R}^3 \to \mathbb{R}^3$

Linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^3$

The inverse of more comlicated transformations can be obtained by composition.

- Translations leave directions unchanged.

Coordinate Transformations

There are two ways we can view transformations. One is an active interpretation, where objects are actively transformed and hence change their position and orientation. The other is a passive interpretation, where we change the coordinate system while all objects remain unchanged. That is similar to the situation in a train. We can either think of a train moving through a resting world or of a train

VI at rest with the world passing by.

In computer graphics both interpretations are used and it is important to understand the one-to-one correspondence between affine transformations and coordinate changes.

Consider a coordinate system with origin $\bar{p} \in \mathbb{R}^3$ and orthonormal basis $\{\bar{u}, \bar{v}, \bar{w}\}$, where the coordinates (u, v, w) describe the point

$$\bar{p} + u\bar{u} + v\bar{v} + w\bar{w}.$$

We can transform (u, v, w) into canonical coordinate system with origin $\bar{0} \in \mathbb{R}^3$ and orthonormal basis $\{\bar{e}_x, \bar{e}_y, \bar{e}_z\}$ by

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \bar{u} & \bar{v} & \bar{w} & \bar{p} \\ & \bar{0}^T & & 1 \end{pmatrix}}_{M} \begin{pmatrix} u \\ v \\ w \\ 1 \end{pmatrix} \qquad \begin{pmatrix} u \\ v \\ w \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \bar{u}^T & \bar{0} \\ \bar{v}^T & \\ \bar{w}^T & \\ \bar{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} & -\bar{p} \\ \bar{0}^T & 1 \end{pmatrix}}_{M^{-1}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$