Exercise 3

1. Explain why it is a good heuristic to choose the variable that is most constrained but the value that is least constraining in a CSP search.

The most constrained variable makes sense because it chooses a variable that is likely to cause an early failure, and it is more efficient to fail as early as possible.
Also called *Fail First heuristic* or *Minimum Remaining Values*.

The least constraining value heuristic makes sense because it allows the most chances for future assignments to avoid a conflict. Once a variable is selected (MRV), increase the chance to find a solution.

2. Explain the main steps of the AC-3 algorithm. Check your approach with an example, e.g., arc consistency of the partial assignment {WA=green, V =red} for the map-colouring problem.

Init: Put all the arcs of the csp in a queue

While the queue is not empty

- Take an arc out of the queue (X_i, X_j)
- Check if a domain value of X_i must be removed
- If a domain value is removed
  - check whether the domain of X_i is empty. If yes, stop
  - Put all arcs of neighbors of X_i in the queue

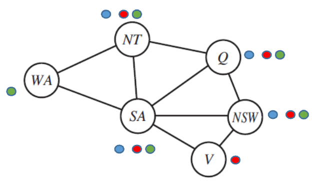
Finished if the queue is empty

function AC-3(csp) **returns** false if an inconsistency is found and true otherwise
inputs: csp, a binary CSP with components (X, D, C)
local variables: $queue$, a queue of arcs, initially all the arcs in csp

while $queue$ is not empty **do**
  $(X_i, X_j) \leftarrow$ REMOVE-FIRST($queue$)
  if REVISE(csp, X_i, X_j) **then**
    if size of $D_i = 0$ **then return** $false$
    for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
      add (X_k, X_i) to $queue$
return $true$

---

function REVISE(csp, X_i, X_j) **returns** true iff we revise the domain of X_i
  $revised \leftarrow false$
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      $revised \leftarrow true$
  return $revised$



Assume initial queue

SA-WA	del	del	del
SA-V	SA-V	NT-WA	NT-SA
NT-WA	NT-WA	NT-SA	NSW-SA
NT-SA	NT-SA	NSW-SA	NSW-V
NSW-SA	NSW-SA	NSW-V	Q-NT
NSW-V	NSW-V	Q-NT	Q-SA
Q-NT	Q-NT	Q-SA	Q-NSW
Q-SA	Q-SA	Q-NSW	V-SA
Q-NSW	Q-NSW	V-SA	WA-SA
	V-SA	WA-SA	SA-NT

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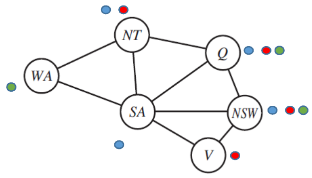

Assume initial queue

SA-WA	del ●	del ●	del ●	del ●
SA-V	SA-V	NT-WA	NT-SA	NSW-SA
NT-WA	NT-WA	NT-SA	NSW-SA	NSW-V
NT-SA	NT-SA	NSW-SA	NSW-V	Q-NT
NSW-SA	NSW-SA	NSW-V	Q-NT	Q-SA
NSW-V	NSW-V	Q-NT	Q-SA	Q-NSW
Q-NT	Q-NT	Q-SA	Q-NSW	V-SA
Q-SA	Q-SA	Q-NSW	V-SA	WA-SA
Q-NSW	Q-NSW	V-SA	WA-SA	SA-NT
	V-SA	WA-SA	SA-NT	WA-NT

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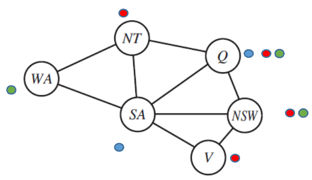

Assume initial queue

....

del ●	del ●	del ●
NSW-SA	NSW-V	Q-NT
NSW-V	Q-NT	Q-SA
Q-NT	Q-SA	Q-NSW
Q-SA	Q-NSW	V-SA
Q-NSW	V-SA	WA-SA
V-SA	WA-SA	SA-NT
WA-SA	SA-NT	WA-NT
SA-NT	WA-NT	V-NSW
WA-NT	V-NSW	SA-NSW
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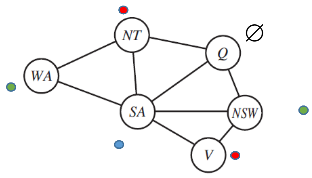

Assume initial queue

....

del •	del •	del •
NSW-SA	NSW-V	Q-NT
NSW-V	Q-NT	Q-SA
Q-NT	Q-SA	Q-NSW
Q-SA	Q-NSW	~~V-SA~~
Q-NSW	V-SA	WA-SA
V-SA	WA-SA	SA-NT
WA-SA	SA-NT	WA-NT
SA-NT	WA-NT	V-NSW
WA-NT	V-NSW	SA-NSW
	SA-NSW	

3. Would it be rational for an agent to hold the three beliefs P(A) = 0.4, P(B) = 0.3 and P(A V B) = 0.5? If so, what range of probabilities would be rational for the agent to hold for A ∧ B? Make up a table and show how it supports your argument about rationality. Then draw another version of the table where P(A V B) = 0.8. Explain whether this is also consistent?

What are the atomic states of this "world"?

Two Boolean variables.

	B=t	B=f
A=t	a	b
A=f	c	d

P(A=t) = a+b = 0.4 → b = 0.4-a

P(B=t) = a+c = 0.3 → c = 0.3-a

P(A=t V B=t) = a+b+c = 0.5 → a+b+c = a+0.4-a+0.3-a=0.5 → -a+0.7=0.5 → a=0.2

a+b+c+d = 1 → d = 0.5 → b=0.2 and c=0.1

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a+b+c+d = 1 → d = 0.5 → b=0.2 and c=0.1

a+b+c = a+0.4-a+0.3-a=0.8 → a= - 0.1

4. For each of the following statements, either proof it is true or give a counterexample.

- If P(a|b,c) = P(b|a,c), then P(a|c) = P(b|c)
- If P(a|b,c) = P(a), then P(b|c) = P(b)
- If P(a|b) = P(a), then P(a|b,c) = P(a|c)

4. For each of the following statements, either proof it is true or give a counterexample.

- If P(a|b,c) = P(b|a,c), then P(a|c) = P(b|c)

P(a | b) = P(a ∧ b) / P(b)

$$P(a|b,c) = \frac{P(a,b,c)}{P(b,c)} \qquad P(b|a,c) = \frac{P(a,b,c)}{P(a,c)}$$

$$\frac{P(a,b,c)}{P(b,c)} = \frac{P(a,b,c)}{P(a,c)}$$

$$\frac{1}{P(b,c)} = \frac{1}{P(a,c)}$$

$$P(b,c) = P(a,c)$$

$$\frac{P(b|c)}{P(c)} = \frac{Pa|c)}{P(c)}$$

$$P(b|c) = P(a|c)$$ → TRUE

4. For each of the following statements, either proof it is true or give a counterexample.

- If P(a|b,c) = P(b|a,c), then P(a|c) = P(b|c) → *TRUE*
- If P(a|b,c) = P(a), then P(b|c) = P(b)
  The statement P(a|b, c) = P(a) merely states that a is independent of b and c, it makes no claim regarding the dependence of b and c.
  Counter-example:
  P(Weather| Catch, Cavity) = P(Weather) but P(Catch| Cavity) ≠ P(Catch) → FALSE
- If P(a|b) = P(a), then P(a|b,c) = P(a|c)
  While the statement P(a|b) = P(a) implies that a is independent of b, it does not imply that a is conditionally independent of b given c.
  Counter-example:
  P(Battery| Gas) = P(Battery) but P(Battery| Gas, Starts) ≠ P(Battery|Starts). → *FALSE*

5. It is quite often useful to consider the effect of some specific propositions in the context of some general background evidence that remains fixed, rather than in the complete absence of information. The following questions ask you to prove more general versions of the product rule, with respect to some background evidence e. Prove the conditionalized version of the general product rule:
   P(A,B|E) = P(A|B,E) P(B|E).

$$P(A,B|E) = \frac{P(A,B,E)}{P(E)}$$

$$P(A|B,E)P(B|E) = \frac{P(A,B,E)}{P(B,E)}\frac{P(B,E)}{P(E)} = \frac{P(A,B,E)}{P(E)}$$

→ P(A,B|E) = P(A|B,E) P(B|E)

6. This exercise investigates the way in which conditional independence relationships affect the amount of information needed for probabilistic calculations.
   a) Suppose we wish to calculate **P**(H| E_1,E_2) and we have no conditional independence information. Which of the following set of numbers are sufficient for the calculation?
      I. **P**(E_1,E_2) , **P**(H), **P**(E_1|H), **P**(E_2|H)
      II. **P**(E_1,E_2) , **P**(H), **P**(E_1, E_2|H)
      III. **P**(H), **P**(E_1|H), **P**(E_2|H)
   b) Suppose we know that **P**(E_1| H, E_2)= **P**(E_1 | H) for all values of *H*, E_1, E_2. Now which of the three sets are sufficient?

$$P(H|E_1,E_2) = \frac{P(E_1,E_2|H)P(H)}{P(E_1,E_2)} \qquad Bayes\ rule\colon\ P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

Clearly II. is sufficient

Intuitively III. is insufficient, because it provides no information about correlations of E_1 and E_2.

Suppose H has *m*, E_1 has *n* and E_2 has *o* possible values.

P(H| E_1,E_2) contains *(m-1)*n*o* independent values

III. has *(m-1)+m*(n-1)+m*(o-1)*

I. has *(n*o-1) +(m-1)+m*(n-1)+m*(o-1)*

$P(H|E_1,E_2)$ contains *(m-1)*n*o* independent values
III. has *(m-1)+m*(n-1)+m*(o-1)*
I. has *(n*o-1) +(m-1)+m*(n-1)+m*(o-1)*

Let m=n=o=4

$P(H|E_1,E_2)$ contains *3*4*4= 48* independent values

III. has *3+12+12=27*

I. has *15+3+12+12=43* independent values

→if *m, n ,o* are large enough, I. and III. are insufficient.

6. This exercise investigates the way in which conditional independence relationships affect the amount of information needed to for probabilistic calculations.
   a) Suppose we wish to calculate $P(H| E_1,E_2)$ and we have no conditional independence information. Which of the following set of numbers are sufficient for the calculation?
      - **I.** $\mathbf{P}(E_1,E_2)$, $\mathbf{P}(H)$, $\mathbf{P}(E_1|H)$, $\mathbf{P}(E_2|H)$
      - **II.** $\mathbf{P}(E_1,E_2)$, $\mathbf{P}(H)$, $\mathbf{P}(E_1, E_2|H)$
      - **III.** $\mathbf{P}(H)$, $\mathbf{P}(E_1|H)$, $\mathbf{P}(E_2|H)$
   b) Suppose we know that $\mathbf{P}(E_1| E_2,H)= \mathbf{P}(E_1 | H)$ for all values of *H, E_1, E_2*. Now which of the three sets are sufficient?

$$P(H|E_1,E_2) = \frac{P(E_1,E_2|H)P(H)}{P(E_1,E_2)}$$

If E_1 and E_2 are conditional independent given H

$$P(H|E_1,E_2) = \frac{P(E_1|H)P(E_2|H)P(H)}{P(E_1,E_2)}$$ I. is sufficient

$$= \alpha P(E_1|H)P(E_2|H)P(H)$$ III. is sufficient

All are sufficient