Assignment 7

Discussed during the tutorial on December 8th, 2022

7.1 (10 points) Suppose that an LP in SEF is changed by defining x'_1 by $x_1 = 10x'_1$ and substituting for x_1. The new problem is equivalent to the old problem. If we now solve the new problem with the simplex method (and apply the same rule for the choice of the leaving variable), will the same choice of entering variables occur as when the old problem was solved? Discuss this question for the four entering variable rules mentioned below:

(a) The largest coefficient rule: When Dantzig first proposed the simplex algorithm, he suggested choosing among all $j \notin B$ with $c_j > 0$, and $k \notin B$ such that c_k is maximum. This rule is sometimes called Dantzig's rule or the largest coefficient rule. Based on the formula for the objective value for the new basic solution, it chooses the variable giving the largest increase in the objective function per unit increase in the value of the entering variable.

(b) Bland's rule.

(c) The largest improvement rule: This is the rule that chooses the entering variable to be the one that leads to the largest increase in the objective value. To choose the variable, therefore, we have to compute for each $j \notin B$, for which $c_j > 0$, the amount t_j by which we will be able to increase the value of x_j, and find the maximum of $c_j t_j$ over all such j and choose the corresponding index k.

(d) The steepest edge rule: This rule is geometrically motivated. In moving from the current basic feasible solution to a new one, we move a certain distance in $\mathbb{R}^n$. This rule chooses the variable which gives the largest increase in the objective value per unit distanced moved. Suppose that $k \notin B$ is chosen and the new value of x_k will be t. Then the change in the value of the basic variable x_t is $-A_{ik}t$. Since there is no change in the value of the other nonbasic variables, the distance moved is

$$\sqrt{t^2 + \sum_{i \in B} (A_{ik}t)^2} \ .$$

Since the change in the objective value is tc_k, we choose $k \notin B$ with $c_k > 0$ so that

$$\frac{c_k}{\sqrt{1 + \sum_{i \in B} (A_{ik})^2}}$$

is maximized.

7.2 (10 points) In this exercise we go over a classical example on which the largest coefficient rule performs $2^n - 1$ iterations. For every $n \geq 2$ consider the LP given below:

$$\max \sum_{j=1}^{n} 10^j x_j \text{ subject to } 10^j x_i + 2 \sum_{j=1}^{i-1} 10^j x_j \leq 10^i, i = 2, \ldots, n; x \geq 0 \ .$$

(a) Prove that for every $n \geq 2$, the above LP has 2^n basic solutions.

(b) Describe an optimal solution for the above family of LPs for every $n \geq 2$, and prove that your solution is the unique optimal solution for each n.

(c) Prove that the simplex method with the largest coefficient rule on the above LP with $n = 2$, starting from the basic feasible solution $\tilde{x} = 0$, visits every basic feasible solution.

(d) Using your result from the previous part and induction, prove that for every $n \geq 2$, on the above family of LP problems, the simplex method with the largest coefficient rule performs $2^n - 1$ iterations to find the optimal solution.

7.3 (10 points) Consider the following LP in SEF:

$$\max\{c^T x \mid Ax = b, x \geq 0\}, \tag{(P)}$$

where A is an $m \times n$ matrix. Let $\bar{x}$ be a feasible solution to (P) and let $J = \{j \mid \bar{x}_j > 0\}$. Call a vector d a *good direction* for $\bar{x}$ if it satisfies the following properties:

- $d_j < 0$ for some $j \in J$,
- $Ad = 0$,
- $d_j = 0$ for all $j \notin J$.

(a) Show that if the columns of A_J are linearly dependent, then there exists a good direction for $\bar{x}$.
Hint: Use the definition of linear dependence to get a vector d. Then possibly replace d by $-d$.

(b) Show that if $\bar{x}$ is not basic, then there exists a good direction for $\bar{x}$.
Hint: Use (a) and Exercise 1.

(c) Show that if $\bar{x}$ has a good direction, then there exists a feasible solution x' of (P) such that the set $J' = \{j \mid x'_j > 0\}$ has fewer elements than J.
Hint: Let $x' = x + \varepsilon d$ for a suitable value $\varepsilon > 0$.

(d) Show that if (P) has a feasible solution, then it has a feasible solution that is basic.
Hint: Use (b) and (c) repeatedly.

(e) Give an algorithm that takes as input a feasible solution for (P) and returns a basic feasible solution for (P).

Upload your solutions as a `.pdf`-file to the course page on the TUHH e-learning portal until 8am on December 6th.