Module 2: Linear Programs (Basis)

Notation

Notation

Consider

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

Notation

Consider

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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$$
\begin{array}{c}
\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\
A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}
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$$

Notation

Let B be a subset of column indices.

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Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

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$$B = \{1, 2, 3\}$$

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$$A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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$$B = \{1, 3, 4\}$$

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Notation

Let B be a subset of column indices.
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$$B = \{5\}$$

Notation

Consider

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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Let B be a subset of column indices.
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$$B = \{5\} \qquad A_{\{5\}} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

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Notation

A_j denotes column j of A.

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Notation

Let B be a subset of column indices.
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$$B = \{5\} \qquad A_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Notation

A_j denotes column j of A.

Basis

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$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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(1) A_B is a square matrix,

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Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

Basis

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Is $B = \{1, 2, 3\}$ a basis?

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Is $B = \{1, 2, 3\}$ a basis? $A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ YES

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Is $B = \{1, 5\}$ a basis?

$$A_B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

NO

A_B is not square

Basis

Consider

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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Basis

Consider

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\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\
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Definition

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(1) A_B is a square matrix,

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Is $B = \{2, 3, 4\}$ a basis?

$$A_B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

NO

columns of A_B are dependent

Question

Does every matrix have a basis?

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$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Theorem

Max number of independent columns $=$
Max number of independent rows.

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Theorem

Max number of independent columns $=$
Max number of independent rows.

Remark

Let A be a matrix with independent rows. Then B is a basis if and only if B is a maximal set of independent columns of A.

Basic Solutions

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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Let B be a basis of A.

Basic Solutions

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Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,

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Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Basic Solutions

$$
\begin{array}{c}
\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\
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\end{array}
x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}
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Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
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Example

Basis $B = \{1, 2, 4\}$.

Basic Solutions

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- if $j \in B$ then x_j is a basic variable,
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Example

Basis $B = \{1, 2, 4\}$. Then

- x_1, x_2, x_4 are the basic variables,

Basic Solutions

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Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Example

Basis $B = \{1, 2, 4\}$. Then

- x_1, x_2, x_4 are the basic variables, and
- x_3, x_5 are the non-basic variables.

Basic Solutions

Basic Solutions

$$\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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Definition

x is a basic solution for basis B if

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Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

Basic Solutions

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Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

Basic Solutions

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$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$: basic sol'n for $B = \{1, 2, 3\}$

Basic Solutions

$$\underbrace{\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \end{array}}_{}$$

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(1) $Ax = b$

Basic Solutions

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(1) $Ax = b$ ✓

Basic Solutions

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$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}: \text{ basic sol'n for } B = \{1, 2, 3\}$$

(1) $Ax = b$ ✓

(2) $x_4 = x_5 = 0$

Basic Solutions

$$\underbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \multicolumn{5}{c}{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$: basic sol'n for $B = \{1, 2, 3\}$

(1) $Ax = b$ ✓

(2) $x_4 = x_5 = 0$ ✓

Basic Solutions

$$
\underbrace{\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}
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Basic Solutions

$$\begin{array}{c}
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Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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Problem

Find a basic solution x for the basis $B = \{1, 4\}$?

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Problem

Find a basic solution x for the basis $B = \{1, 4\}$?

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} x$$

$$\begin{array}{c} \begin{array}{cccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} \end{array} \\ \underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{b}$$

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$\Rightarrow$

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

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$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$
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Thus, the basic solution is $x = (4,\ 0,\ 0,\ 2)^\top$.

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Question

Did we have a choice for a basic solution x given $B = \{1, 4\}$?

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\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A}
\end{array} x = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{b}$$

Problem

Find a basic solution x for the basis $B = \{1, 4\}$?

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$$= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{x_2}_{=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{x_3}_{=0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

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Thus, the basic solution is $x = (4,\ 0,\ 0,\ 2)^\top$.

Question

Did we have a choice for a basic solution x given $B = \{1, 4\}$? NO!

Basic Solutions – Uniqueness

Proposition

Consider $Ax = b$ and a basis B of A.
Then there exists a unique basic solution x for B.

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For $B = \{1, 2, 4\}$,

$$A_B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \quad \text{(basic variables)}$$

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columns of A_B and elements of x_B are ordered by B!

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Since B is a basis, it implies A_B is non-singular, i.e., A_B^{-1} exists.

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$$

Since B is a basis, it implies A_B is non-singular, i.e., A_B^{-1} exists.

Hence, $x_B = A_B^{-1} b$.

When Is a Vector Basic?

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Question

Is $x = (0, 0, 3, 0, 3)^\top$ basic?

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$$\underbrace{\begin{pmatrix} 3 & 2 & 1 & 4 & 1 \\ -1 & 1 & 0 & 2 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}_{b}$$

Question

Is $x = (0, 1, 0, 1, 0)^\top$ basic? NO!

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is a column submatrix of A_B.

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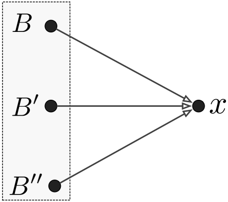

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Definition

A basic solution x of $Ax = b$ is feasible if $x \geq 0$, i.e., if it is feasible for (P).

Consider the system

$$Ax = b$$

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(6) A basic solution is feasible if it is non-negative.