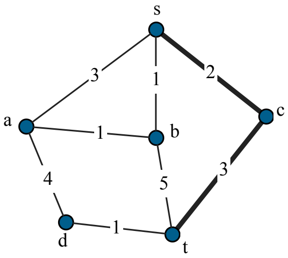
Module 3: Duality through examples (Shortest Path Algorithm)

Recap: Feasible Widths via Duality

The figure on the right shows another simple instance of the shortest s, t-path problem.



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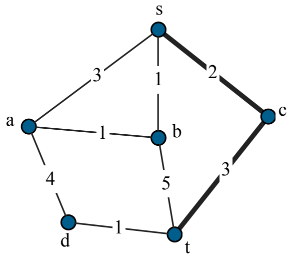

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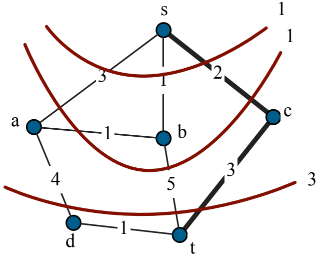

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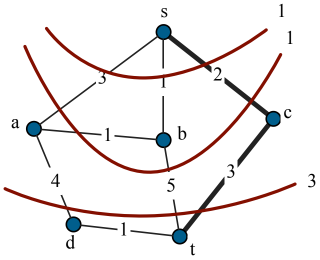

Shortest path LP:

$$\min \quad \sum (x_e \,:\, e \in E)$$

$$\text{s.t.} \quad \sum (x_e \,:\, e \in \delta(S)) \geq 1$$

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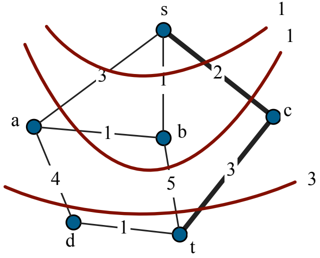

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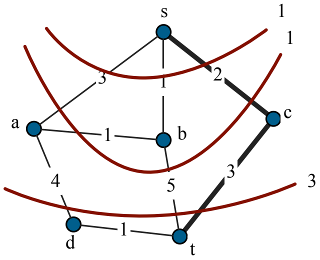

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Letting

$$x_e = \begin{cases} 1 & e \textbf{ bold} \text{ in figure} \\ 0 & \text{otherwise} \end{cases}$$

for all $e \in E$ is feasible for shortest path LP.

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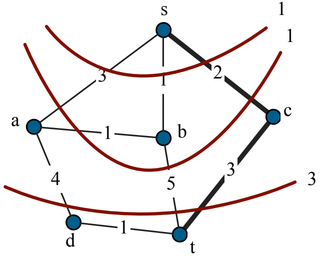

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Letting

$$y_{\{s\}} = y_{\{s,b\}} = 1, \; y_{\{s,a,b,c\}} = 3,$$

and $y_S = 0$ for all other s,t-cuts $\delta(S)$ yields a feasible dual solution of value 5!

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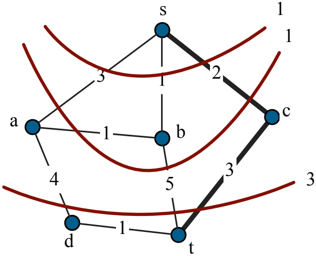

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If $\bar{x}$ is feasible for shortest path LP, and $\bar{y}$ is feasible for its dual then $b^T\bar{y} \leq c^T\bar{x}$.

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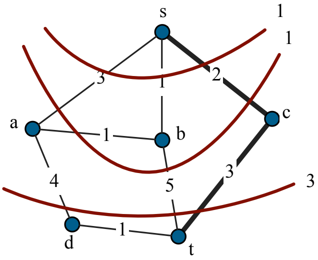

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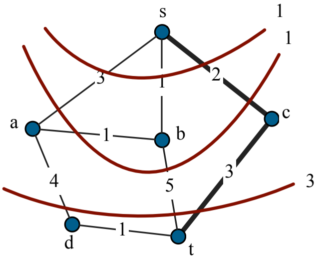

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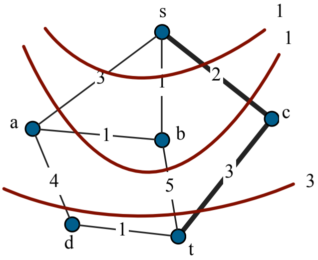

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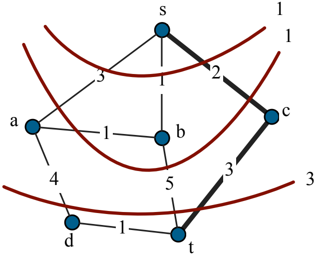

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Today:

1. How did we find the bold path?
2. How did we find the dual solution?
3. Is there always a shortest s,t-path and a dual solution whose value matches its length?

An Algorithm for the Shortest s, t-Path Problem

Arcs and Directed Paths

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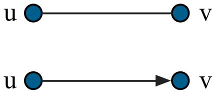

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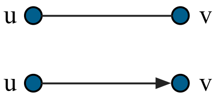

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A directed path is then a sequence of arcs:

$$\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{k-1}v_k},$$

where $\overrightarrow{v_iv_{i+1}}$ is an arc in the given graph, and $v_i \neq v_j$ for all $i \neq j$.

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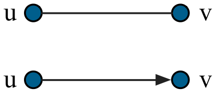

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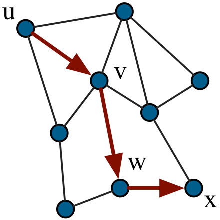


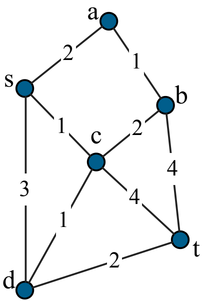
Example:

$$\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{wx}$$

is a directed u, x-path.

Shortest Paths: Algorithmic Ideas

Idea: Find an s,t-path P and a feasible dual y s.t. $c(P) = \mathbb{1}^T y$. How?



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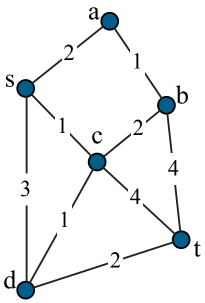

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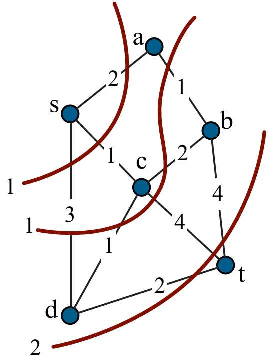

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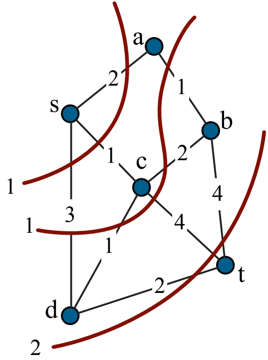

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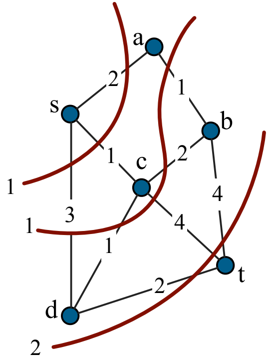

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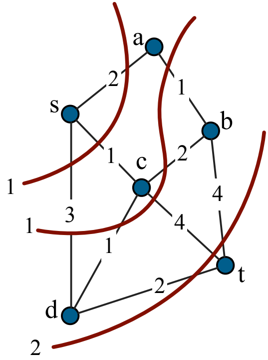

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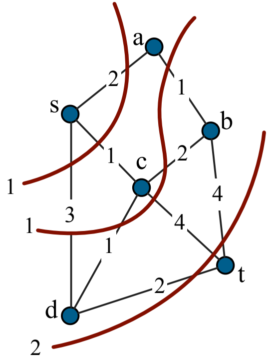

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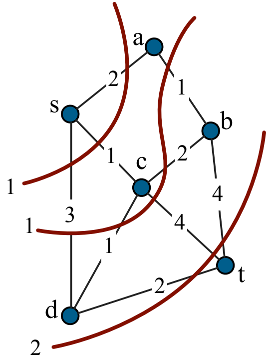

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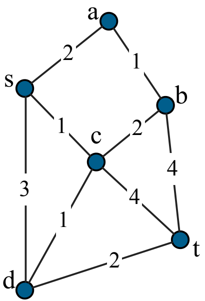

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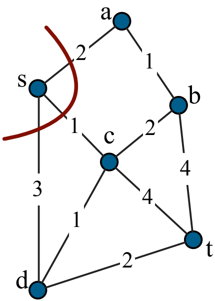

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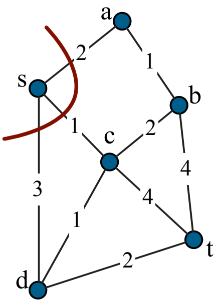

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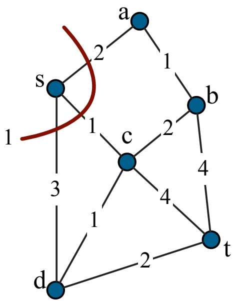

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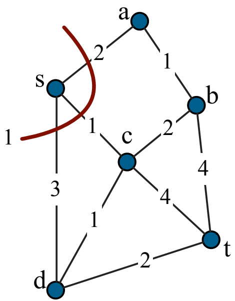

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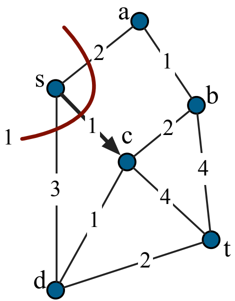
$\longrightarrow$ replace sc by $\overrightarrow{sc}$



$$\begin{aligned} \max \quad & \sum (y_S \,:\, \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \le c_e \\ & \qquad (e \in E) \\ & y \ge \mathbb{0} \end{aligned}$$

Shortest Paths: Building Duals Incrementally

Start with the trivial dual $y = \mathbb{0}$

Simplest s,t-cut: $\delta(\{s\})$
$\longrightarrow$ increase $y_{\{s\}}$ as much as we can maintaining feasibility
$\longrightarrow$ $y_{\{s\}} = 1$

Note: This decreases the slack of sc to 0!
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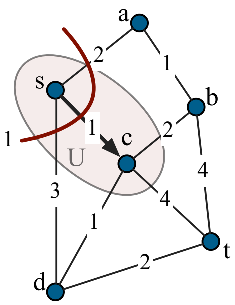
Next: Look at all vertices that are reachable from s via directed paths:

$$U = \{s, c\}$$



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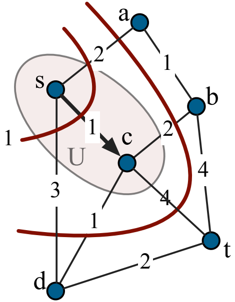

$$\max \quad \sum(y_S : \delta(S) \ s,t\text{-cut})$$

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Shortest Paths: Building Duals Incrementally

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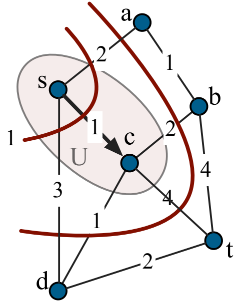
Q: By how much can you increase y_U?



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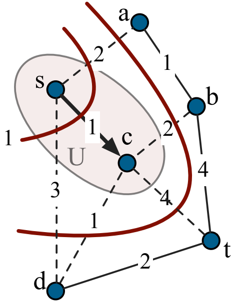
Shortest Paths: Building Duals Incrementally

Q: By how much can you increase y_U? The

maximum increase possible for $y_{\{s,c\}}$ is determined by the slack of edges in $\delta(\{s,c\})$!



$$\max \quad \sum (y_S : \delta(S) \; s,t\text{-cut})$$

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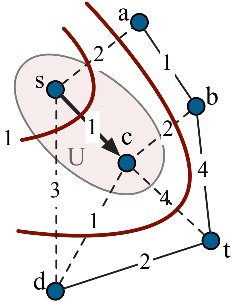

$$\text{slack}_y(sa) =$$
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$$\text{slack}_y(sd) =$$

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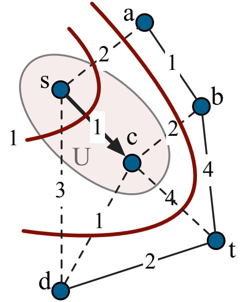
$$\begin{aligned}
\text{slack}_y(sa) &= 2 - 1 = 1 \\
\text{slack}_y(cb) &= \\
\text{slack}_y(ct) &= \\
\text{slack}_y(cd) &= \\
\text{slack}_y(sd) &=
\end{aligned}$$



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Shortest Paths: Building Duals Incrementally

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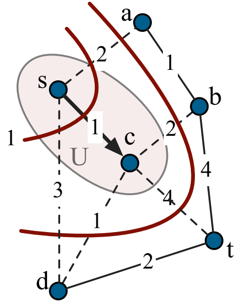

$$\begin{aligned}
\mathsf{slack}_y(sa) &= 2 - 1 = 1\\
\mathsf{slack}_y(cb) &= 2\\
\mathsf{slack}_y(ct) &=\\
\mathsf{slack}_y(cd) &=\\
\mathsf{slack}_y(sd) &=
\end{aligned}$$

$$\begin{aligned}
\max \quad & \sum (y_S : \delta(S)\ s,t\text{-cut})\\
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Shortest Paths: Building Duals Incrementally

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$$\begin{aligned}
\text{slack}_y(sa) &= 2 - 1 = 1 \\
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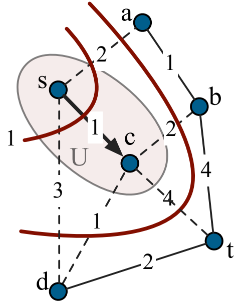
\text{slack}_y(ct) &= 4 \\
\text{slack}_y(cd) &= \\
\text{slack}_y(sd) &=
\end{aligned}$$



$$\max \quad \sum(y_S : \delta(S)\ s,t\text{-cut})$$

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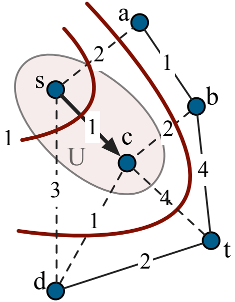

$$
\begin{aligned}
\mathsf{slack}_y(sa) &= 2 - 1 = 1 \\
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$$

$$
\begin{aligned}
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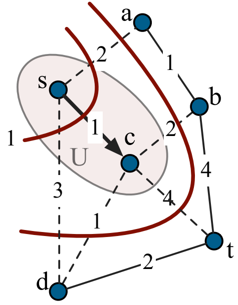
& y \ge \mathbb{0}
\end{aligned}
$$

Shortest Paths: Building Duals Incrementally

Q: By how much can you increase y_U? The

maximum increase possible for $y_{\{s,c\}}$ is determined by the slack of edges in $\delta(\{s,c\})$!

$$\begin{aligned}
\mathsf{slack}_y(sa) &= 2-1=1\\
\mathsf{slack}_y(cb) &= 2\\
\mathsf{slack}_y(ct) &= 4\\
\mathsf{slack}_y(cd) &= 1\\
\mathsf{slack}_y(sd) &= 3-1=2
\end{aligned}$$



$$\begin{aligned}
\max \quad & \sum(y_S \,:\, \delta(S)\ s,t\text{-cut})\\
\text{s.t.} \quad & \sum(y_S \,:\, e\in\delta(S)) \le c_e\\
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& y \ge \mathbb{0}
\end{aligned}$$

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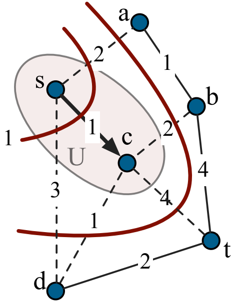
\end{aligned}$$

Edges cd and sa minimize slack. Pick one arbitrarily: sa.



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Shortest Paths: Building Duals Incrementally

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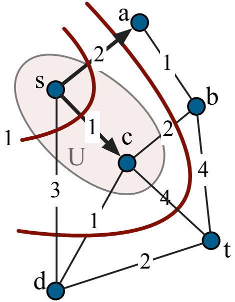

$$\begin{aligned} \mathsf{slack}_y(sa) &= 2-1=1 \\ \mathsf{slack}_y(cb) &= 2 \\ \mathsf{slack}_y(ct) &= 4 \\ \mathsf{slack}_y(cd) &= 1 \\ \mathsf{slack}_y(sd) &= 3-1=2 \end{aligned}$$

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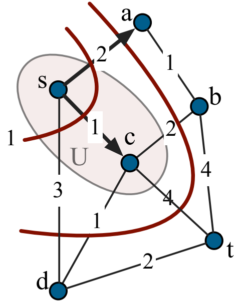
Edges cd and sa minimize slack. Pick one arbitrarily: sa.
Set $y_U = \mathsf{slack}_y(sa) = 1$ and convert sa into arc $\overrightarrow{sa}$

Shortest Paths: Building Duals Incrementally

Q: Which vertices are reachable from s via directed paths?



$$\max \quad \sum(y_S \,:\, \delta(S)\ s,t\text{-cut})$$

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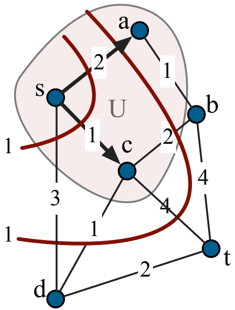
Shortest Paths: Building Duals Incrementally

Q: Which vertices are reachable from s via directed paths?

$$U = \{s, a, c\}$$



$$\max \quad \sum (y_S \,:\, \delta(S) \; s,t\text{-cut})$$

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Shortest Paths: Building Duals Incrementally

Q: Which vertices are reachable from s via directed paths?

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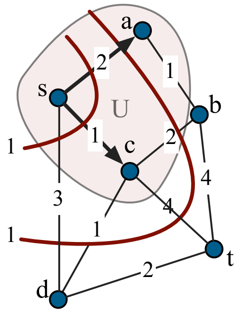
Natural idea: Increase $y_{\{s,a,c\}}$ by as much as we can. How much?



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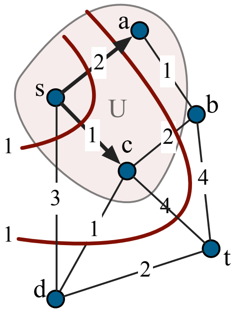
Natural idea: Increase $y_{\{s,a,c\}}$ by as much as we can. How much?
$\longrightarrow$ the slack of cd is 0, and hence

$$y_{\{s,a,c\}} = 0$$



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Shortest Paths: Building Duals Incrementally

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Natural idea: Increase $y_{\{s,a,c\}}$ by as much as we can. How much?
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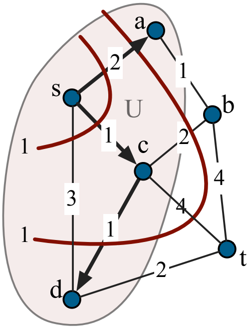
Also: change cd into $\overrightarrow{cd}$, and let

$$U = \{s, a, c, d\}$$

be the reachable vertices from s



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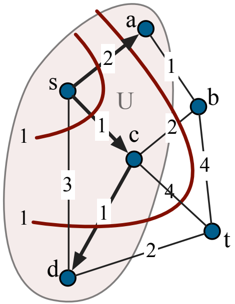
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, c, d\}$$



$$\max \quad \sum (y_S \,:\, \delta(S)\ s,t\text{-cut})$$

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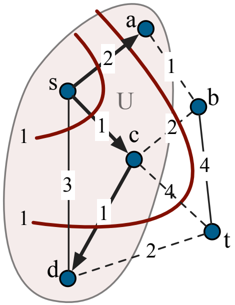
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned} \mathsf{slack}_y(ab) &= \\ \mathsf{slack}_y(cb) &= \\ \mathsf{slack}_y(ct) &= \\ \mathsf{slack}_y(dt) &= \end{aligned}$$



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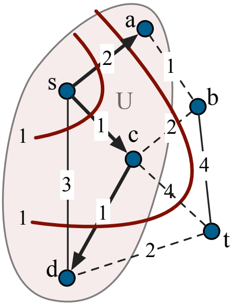
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned} \mathsf{slack}_y(ab) &= 1 \\ \mathsf{slack}_y(cb) &= \\ \mathsf{slack}_y(ct) &= \\ \mathsf{slack}_y(dt) &= \end{aligned}$$



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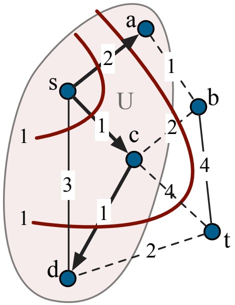
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned} \text{slack}_y(ab) &= 1 \\ \text{slack}_y(cb) &= 2 - 1 = 1 \\ \text{slack}_y(ct) &= \\ \text{slack}_y(dt) &= \end{aligned}$$



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$$y \geq \mathbb{0}$$

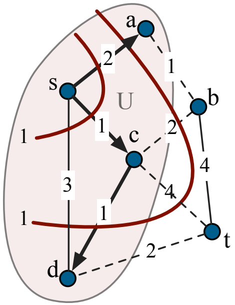
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned}
\mathsf{slack}_y(ab) &= 1 \\
\mathsf{slack}_y(cb) &= 2 - 1 = 1 \\
\mathsf{slack}_y(ct) &= 4 - 1 = 3 \\
\mathsf{slack}_y(dt) &=
\end{aligned}$$



$$\begin{aligned}
\max \quad & \sum (y_S \,:\, \delta(S) \ s,t\text{-cut}) \\
\text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \leq c_e \\
& \qquad (e \in E) \\
& y \geq \mathbb{0}
\end{aligned}$$

Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, c, d\}$$

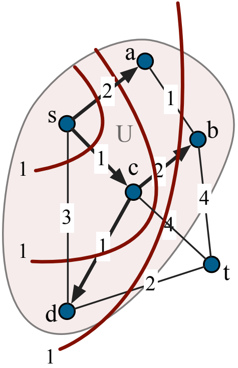
Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned}
\mathsf{slack}_y(ab) &= 1 \\
\mathsf{slack}_y(cb) &= 2 - 1 = 1 \\
\mathsf{slack}_y(ct) &= 4 - 1 = 3 \\
\mathsf{slack}_y(dt) &= 2
\end{aligned}$$

Let $y_{\{s,a,c,d\}} = 1$, add equality arc $\overrightarrow{cb}$, and update the set

$$U = \{s, a, b, c, d\}$$

of vertices reachable from s



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\end{aligned}$$

Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

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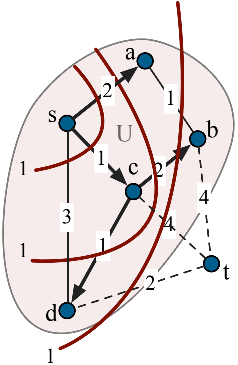


$$\max \quad \sum (y_S \,:\, \delta(S)\ s,t\text{-cut})$$

$$\text{s.t.} \quad \sum (y_S \,:\, e \in \delta(S)) \leq c_e \quad (e \in E)$$

$$y \geq \mathbb{0}$$

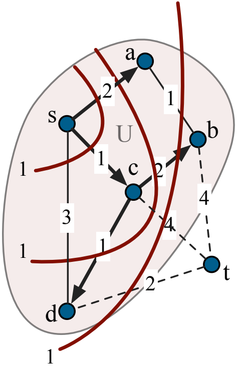
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, b, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned} \mathsf{slack}_y(bt) &= \\ \mathsf{slack}_y(ct) &= \\ \mathsf{slack}_y(dt) &= \end{aligned}$$



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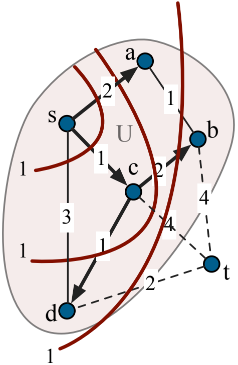
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, b, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned}
\mathsf{slack}_y(bt) &= 4 \\
\mathsf{slack}_y(ct) &= \\
\mathsf{slack}_y(dt) &=
\end{aligned}$$



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Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, b, c, d\}$$

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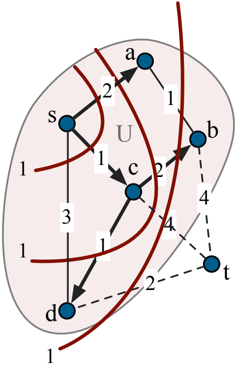

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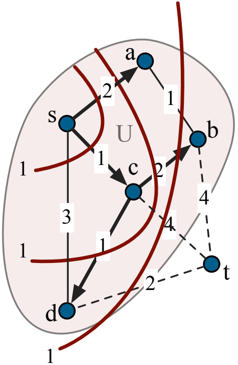

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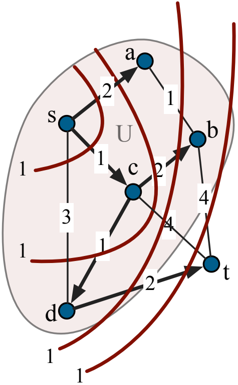
Let $y_{\{s,a,b,c,d\}} = 1$, add equality arc $\overrightarrow{dt}$.



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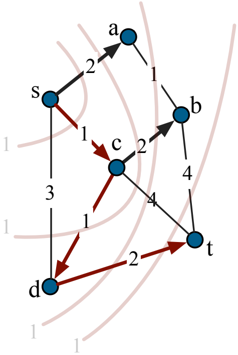
Shortest Paths: Building Duals Incrementally

Note: we now have a directed s,t-path in our graph:

$$P = \overrightarrow{sc}, \overrightarrow{cd}, \overrightarrow{dt},$$

and its length is 4!



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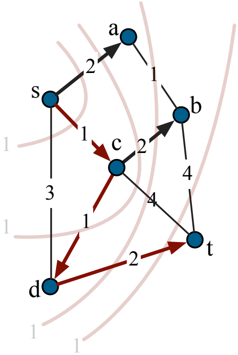
and $y_U = 0$ otherwise.



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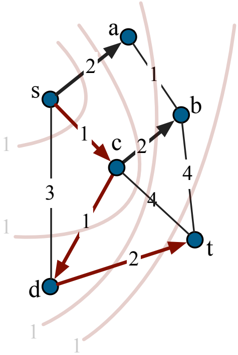

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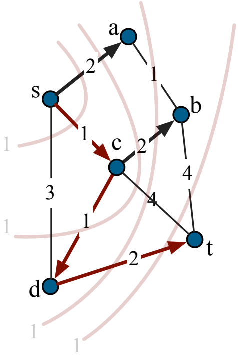
$\longrightarrow$ Path P is a shortest path!



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Shortest Path Algorithm

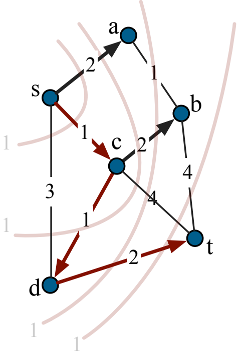
To compute the shortest Path for the instance on the right, we used the following algorithm:

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

$y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$
while $t \notin U$ **do**
  Let ab be an edge in $\delta(U)$ of smallest slack for y where $a \in U$, $b \notin U$
  $y_U := \text{slack}_y(ab)$
  $U := U \cup \{b\}$
  change edge ab into an arc $\overrightarrow{ab}$
end while
return A directed st-path P.



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  (b) a feasible solution y for the dual of the shortest path LP

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- Have a look at the book. It has another full example run of the shortest path algorithm