Module 5: Integer Programs (Cutting Planes)

Overview

In this lecture, we will:

Investigate a class of algorithms known as cutting planes.

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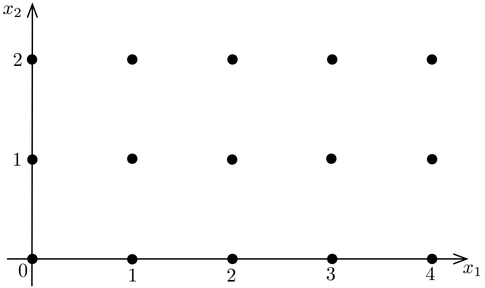

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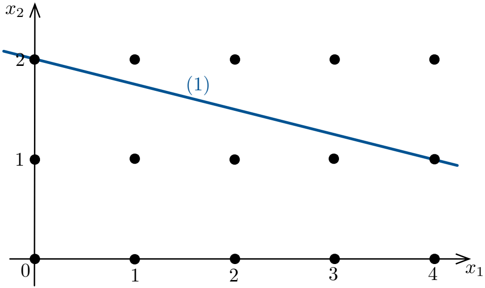

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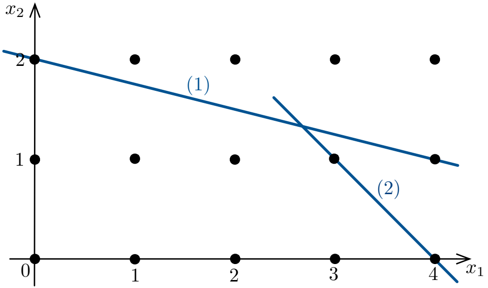

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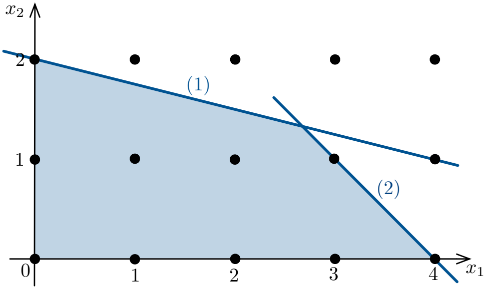

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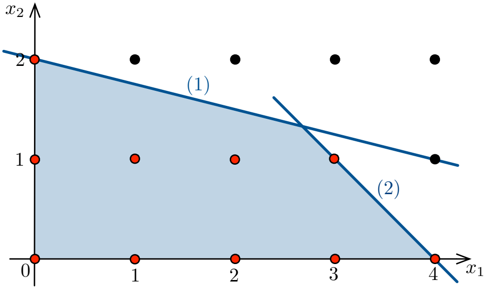

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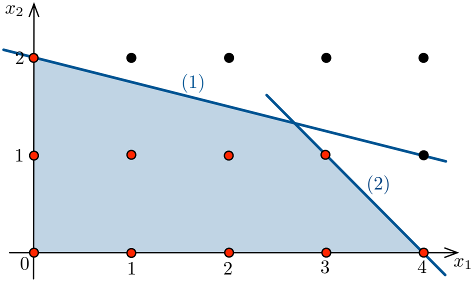

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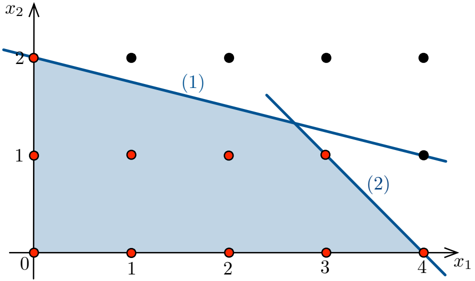

Idea

Solve the LP relaxation instead of the original IP.

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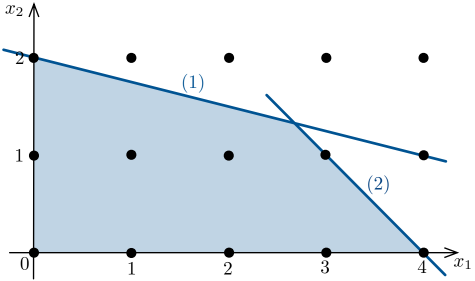

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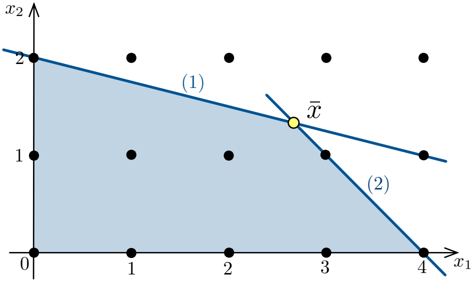

Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal.

$$\max \; \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

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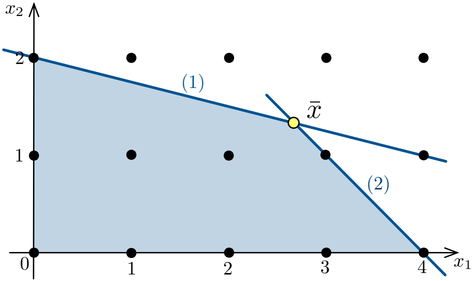

Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. Not integer!

$$
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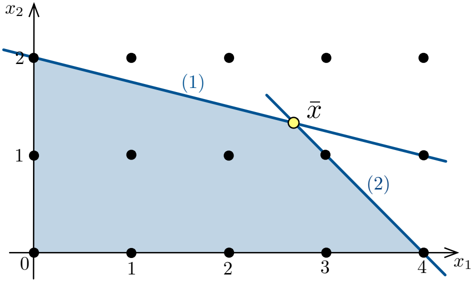

Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. NOT INTEGER!

We now search for a constraint $\alpha^\top x \le \beta$ that

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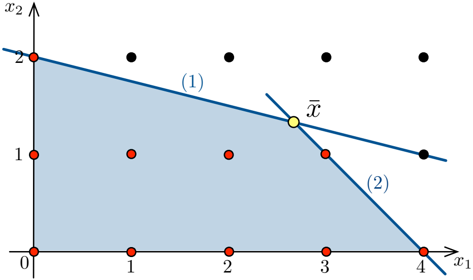

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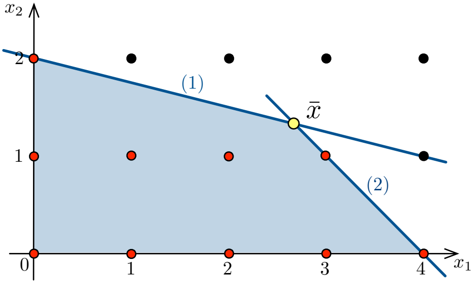

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We now search for a constraint $\alpha^\top x \leq \beta$ that

- is satisfied for all feasible solutions to the IP, and
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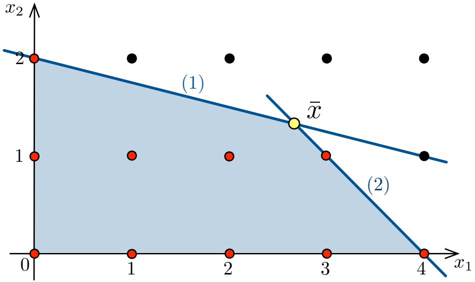

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- is satisfied for all feasible solutions to the IP, and
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We will call this constraint a cutting plane for $\bar{x}$.

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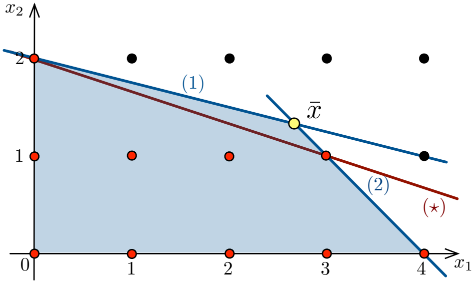

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Example:

$$x_1 + 3x_2 \leq 6. \qquad (\star)$$

After adding $(\star)$ to our relaxation, we get

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$$\max \begin{pmatrix}2 & 5\end{pmatrix} x$$

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$$\begin{pmatrix}1 & 4\\ 1 & 1\\ \color{red}{1} & \color{red}{3}\end{pmatrix} x \le \begin{pmatrix}8\\ 4\\ \color{red}{6}\end{pmatrix} \quad \begin{matrix}(1)\\ (2)\\ \color{red}{(3)}\end{matrix}$$

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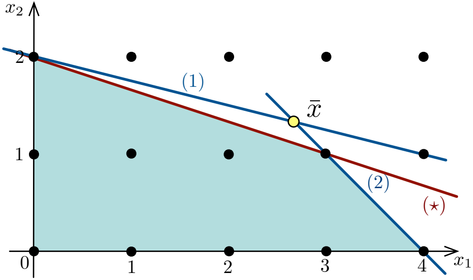

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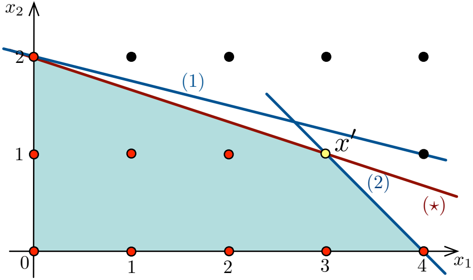

Using Simplex, we get: $x' = (3,1)^\top$ is optimal.

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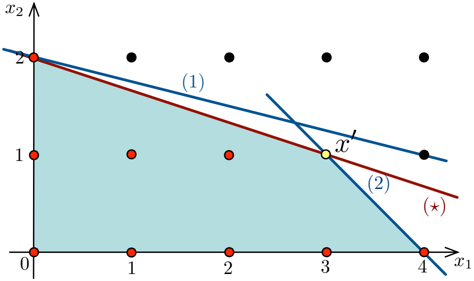

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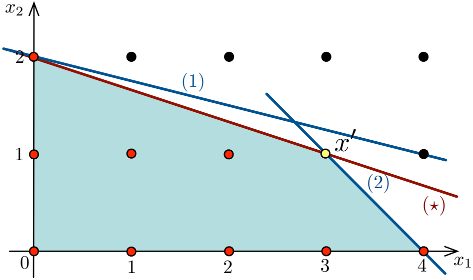

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Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

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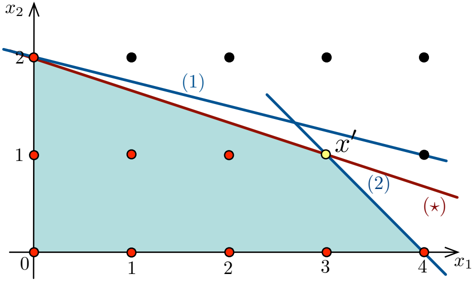

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We have now solved our first IP.

Cutting Plane Scheme

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$$\max\left\{c^\top x : Ax \le b, x \text{ integer}\right\} \qquad \text{(IP)}$$

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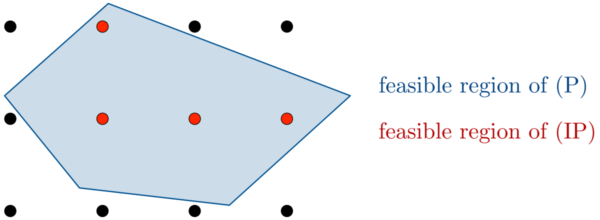

- Let (P) denote $\max\{c^\top x : Ax \le b\}$.

Cutting Plane Scheme

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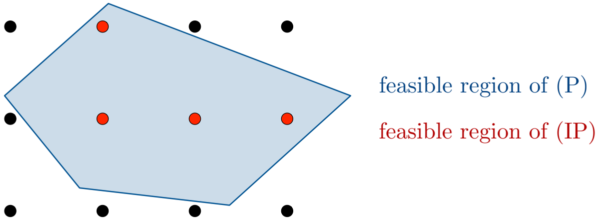

- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
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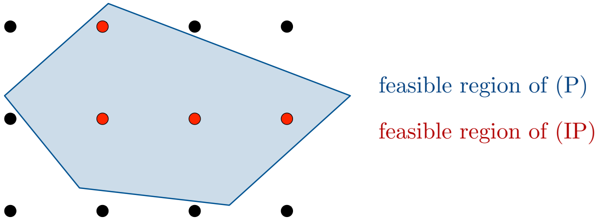

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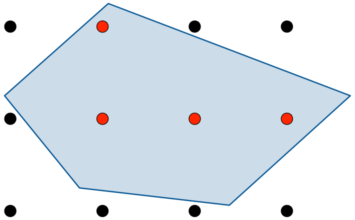

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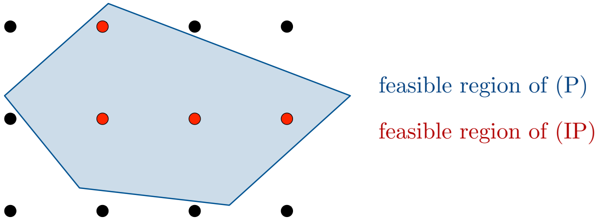

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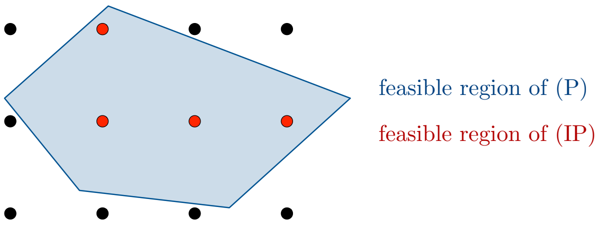

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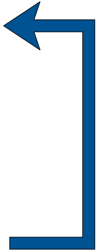
- If $\bar{x}$ is integral, then STOP. $\bar{x}$ is also optimal for (IP).
- Find a cutting plane $a^\top x \le \beta$ for $\bar{x}$.
- Add constraint $a^\top x \le \beta$ to the system $Ax \le b$.



Question

How can we find cutting planes?

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SIMPLEX DOES THIS FOR US!

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Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

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Definition

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Example

$$\lfloor 3.7 \rfloor \quad = \quad 3$$

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\begin{aligned} \lfloor 3.7 \rfloor &= 3 \\ \lfloor 62 \rfloor &= 62 \end{aligned}$$

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\begin{aligned} \lfloor 3.7 \rfloor &= 3 \\ \lfloor 62 \rfloor &= 62 \\ \lfloor -2.1 \rfloor &= -3 \end{aligned}$$

$$\max \; \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$x \geq \mathbf{0}, \; x \text{ integer}$$

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Add a slack variable, $x_3 \geq 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

$$
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\end{array}
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Add a slack variable, $x_3 \ge 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \ge 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

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Add another slack variable, $x_4 \geq 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

Since x_1, x_2 are integers, $x_3 = 8 - x_1 - 4x_2$ and $x_4 = 4 - x_1 - x_2$ are integers.

$$
\begin{array}{|l|}
\hline
\max \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
\qquad x \ge \mathbf{0},\ x \text{ integer} \\
\hline
\end{array}
$$

Add a slack variable, $x_3 \ge 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \ge 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

Since x_1, x_2 are integers, $x_3 = 8 - x_1 - 4x_2$ and $x_4 = 4 - x_1 - x_2$ are integers.

Thus, we can rewrite the IP as

$$
\begin{array}{|l|}
\hline
\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\
\qquad x \ge \mathbf{0},\ x \text{ integer} \\
\hline
\end{array}
$$

Solving the IP

$$\max \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x$$

s. t.

$$\begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\ 4\end{pmatrix}$$

$$x \geq \mathbf{0},\ x \text{ integer}$$

Solving the IP

$$
\begin{array}{l}
\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\
\qquad x \geq \mathbf{0},\ x \text{ integer}
\end{array}
$$

We will now relax the integer program.

Solving the IP

$$\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Solving the IP

$$\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

We will use the Simplex algorithm to solve this.

Solving the IP

$$\begin{array}{l} \max \ \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

Solving the IP

$$\begin{aligned} &\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{aligned} &\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$

Solving the IP

$$\begin{array}{ll} \max & \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Solving the IP

$$\begin{aligned}\max\ & \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x \\ \text{s. t.}\ & \\ & \begin{pmatrix}1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8 \\ 4\end{pmatrix} \\ & x \geq \mathbf{0}\end{aligned}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{aligned}\max\ & \begin{pmatrix}0 & 0 & -1 & -1\end{pmatrix} x + 12 \\ \text{s. t.}\ & \\ & \begin{pmatrix}1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3\end{pmatrix} \\ & x \geq \mathbf{0}\end{aligned}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$. NOT INTEGER

Solving the IP

$$\begin{array}{l} \max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$. NOT INTEGER

Let us use the canonical form to get a cutting plane for $\bar{x}$.

$$
\begin{array}{l}
\max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

$$
\begin{array}{l}
\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{8}{3}$$

$$
\begin{array}{|l|}
\hline
\max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0} \\
\hline
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$
\begin{array}{l}
\max \; \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$
\boxed{\begin{array}{l}
\max \; \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

$$\boxed{\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

For every feasible solution to the IP, $x_1 - x_3 + x_4$ is integer.

$$\begin{array}{|l|}\hline \max \begin{pmatrix}0 & 0 & -1 & -1\end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \\ \hline \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

For every feasible solution to the IP, $x_1 - x_3 + x_4$ is integer.

Hence, every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq \left\lfloor \frac{8}{3} \right\rfloor = 2$$

$$
\begin{array}{l}
\max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

$$
\begin{array}{l}
\max \; \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$
\boxed{\begin{array}{l}
\max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \qquad (\star)$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

$$
\begin{array}{l}
\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

$\Rightarrow$ ($\star$) is a cutting plane for $\bar{x}$.

$$\boxed{\begin{array}{l} \max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

➡ $(\star)$ is a cutting plane for $\bar{x}$.

We can rewrite $(\star)$ as

$$x_1 - x_3 + x_4 + x_5 = 2 \qquad \text{where} \qquad x_5 \geq 0.$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

($\star$) is a cutting plane for $\bar{x}$.

We can rewrite ($\star$) as

$$x_1 - x_3 + x_4 + x_5 = 2 \qquad \text{where} \qquad x_5 \geq 0.$$

We now add this to the relaxation.

$$\begin{aligned}
\max \quad & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} \quad & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}$$

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Solve this using the Simplex algorithm.

$$
\begin{array}{ll}
\max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} x + 11 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} x + 11 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$.

$$\begin{array}{l} \max \quad \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$\begin{array}{l} \max \quad \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} x + 11 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$. INTEGER!

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} x + 11 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

$(3,1,1,0,0)^\top$ is optimal for

$$\begin{aligned}
\max\ & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\
\text{s. t.}\ & \\
& \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\
& x \geq \mathbf{0},\quad x \text{ integer}
\end{aligned}$$

$(3,1,1,0,0)^\top$ is optimal for

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}, \quad x \text{ integer}
\end{array}
$$

$\Rightarrow$

$(3,1)^\top$ is optimal for

$$
\begin{array}{ll}
\max & \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
& x \geq \mathbf{0}, \ x \text{ integer}
\end{array}
$$

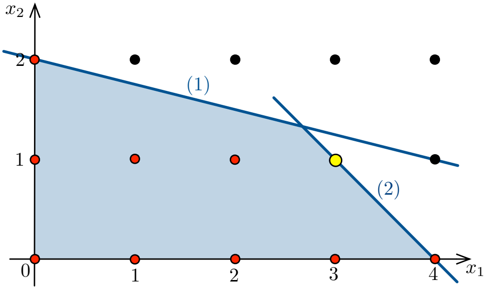
$(3,1,1,0,0)^\top$ is optimal for

$$\begin{array}{l} \max \ \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\ \qquad x \geq \mathbf{0}, \quad x \text{ integer} \end{array}$$

$\Rightarrow$

$(3,1)^\top$ is optimal for

$$\begin{array}{l} \max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ \qquad x \geq \mathbf{0}, \ x \text{ integer} \end{array}$$



Getting Cutting Planes in General

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$
\boxed{
\begin{aligned}
&\max \quad \bar{c}^\top x + \bar{z} \\
&\text{s. t.} \\
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$\Rightarrow$ ($\star$) is a cutting plane for $\bar{x}$.

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- Combine it with a divide and conquer strategy (branch and bound).

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