Module 2: Linear Programs (Standard Equality Forms)

The Key Definition

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3. all other constraints are equality constraints.

$$\max \quad (1, -2, 4, -4, 0, 0)x + 3$$

s.t.

$$\begin{pmatrix} 1 & 5 & 3 & -3 & 0 & -1 \\ 2 & -1 & 2 & -2 & 1 & 0 \\ 1 & 2 & -1 & 1 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Question

Is the following LP in SEF?

$$
\begin{array}{ll}
\max & x_1 + x_2 + 17 \\
\text{s.t.} & \\
& x_1 - x_2 = 0 \\
& x_1 \geq 0
\end{array}
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NO! There is no constraint $x_2 \geq 0$.

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Remarks

- $x_2 \geq 0$ is implied by the constraints.
- x_2 is still free since $x_2 \geq 0$ is not given explicitly.

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1. Find an "equivalent" LP in SEF.

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1. Find an "equivalent" LP in SEF.
2. Solve the "equivalent" LP using Simplex.

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1. Find an "equivalent" LP in SEF.
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What do we mean by equivalent?

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Every LP is equivalent to an LP in SEF.

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Every LP is equivalent to an LP in SEF.

We will illustrate the proof with a series of examples.

Dealing with Minimization

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$$\min \quad (1, 2, -4)(x_1, x_2, x_3)^\top$$

s.t.

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Free Variables

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$$x_1, x_2 \geq 0,\ x_3 \text{ is free.}$$

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Any number is the difference between two non-negative numbers.

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Set $x_3 := a - b$ where $a, b \geq 0$.

Free Variables – Rewrite the Objective Function

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Set $x_3 := a - b$ where $a, b \geq 0$.

$$\begin{aligned} z &= (1,2,3)(x_1, x_2, x_3)^\top \\ &= x_1 + 2x_2 + 3x_3 \end{aligned}$$

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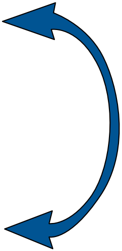

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2. We defined what it means for two LPs to be equivalent.
3. We showed how to convert any LP into an equivalent LP in SEF.
4. To solve any LP, it suffices to know how to solve LPs in SEF.