

So far ...

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$$\min c^T x$$
s.t. $Ax \ge b$

$$x \ge 0$$

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- Integer linear programs.

Both have linear/affine constraints.

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So far ...

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- Integer linear programs.

Both have linear/affine constraints.

Now: Nonlinear generalization!

$$\min c^T x$$
s.t. $Ax \ge b$

$$x \ge 0$$

$$x \text{ integer}$$

A non-linear program (NLP) is of the form

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\begin{aligned} & \min & f(x) \\ & \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & & \cdots \\ & g_m(x) \leq 0, \end{aligned}
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where

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where

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$$\min f(x)$$
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$$\dots$$

$$g_m(x) < 0,$$

Note: Linear programs are NLPs!

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- $x \in \mathbb{R}^n$,
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Example 1: Finding Close Points in an LP

Problem: we are given an LP (P), and an infeasible point \bar{x} .

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 $\label{eq:Goal: find a point } \text{Goal: find a point } x \in P \text{ that is as close as possible to } \bar{x}.$

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e.g.: find a point $x \in P$ that minimizes the Euclidean distance to \bar{x} :

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$$||x - \bar{x}||_2 = \sqrt{\sum_{i=1}^n (x_i - \bar{x}_i)^2}$$

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$$\min \|x - \bar{x}\|_2$$

s.t. $x \in P$

Example 2: Binary IP via NLP

Suppose we are given a binary IP (i.e., an integer program all of whose variables are binary).

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \\ & x_j \in \{0,1\} \quad (j \in \{1,\dots,n\}) \end{array}$$

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Recall: (binary) IPs are generally hard to solve!

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$$x_j(1-x_j) = 0 \quad (j \in [n]) \quad (\star)$$

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Correctness: For $j \in [n]$, (*) is holds iff $x_j = 0$ or $x_j = 1$.

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Q: Can you change the NLP to express the fact that x_j is any non-negative integer instead of binary?

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Ideas?

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq \mathbb{0} \\ & \sin(\pi \, x_j) = 0 \quad (j \in [n]) \end{array} \tag{*}$$

Correctness: For $j \in [n]$, (\star) is holds iff $x_j = 0$ or $x_j = 1$.

Q: Can you change the NLP to express the fact that x_j is any non-negative integer instead of binary?

Correctness: note that $\sin(\pi x_j) = 0$ only if x_j is an integer.

Example 3: Fermat's Last Theorem

Conjecture [Fermat, 1637]

There are no integers $x,\,y,\,z\geq 1$ and $n\geq 3$ such that

$$x^n + y^n = z^n.$$



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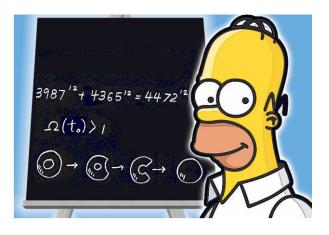
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 s.t.
$$x_i \geq 1 \quad (i = 1 \dots 3)$$
$$x_4 \geq 3$$

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- In fact, the value of a solution (x_1, x_2, x_3, x_4) is 0 iff

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 - $\sin \pi x_i = 0$, for all $i = 1 \dots 3$.

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Remark

Fermat's Last Theorem is true iff the NLP has optimal value greater than 0.

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Proving Fermat's Last Theorem amounts to showing that the value 0 can not be attained!

Recap

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- Non-linear programs are strictly more general than integer programs, and thus likely difficult to solve.
- Some famous questions in Math can easily be reduced to solving certain NLPs