Module 2: Linear Programs (Certificates)

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

Questions

Consider a linear program.

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

Questions

Consider a linear program.

- If it is infeasible, how can we prove it?

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

Questions

Consider a linear program.

- If it is infeasible, how can we prove it?
- If we have an optimal solution, how can we prove it is optimal?

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

Questions

Consider a linear program.

- If it is infeasible, how can we prove it?
- If we have an optimal solution, how can we prove it is optimal?
- If it is unbounded, how can we prove it?

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

Questions

Consider a linear program.

- If it is infeasible, how can we prove it?
- If we have an optimal solution, how can we prove it is optimal?
- If it is unbounded, how can we prove it?

This can be always be done!

Proving Infeasibility

Proving Infeasibility

The following linear program is infeasible:

$$\max \quad (3, 4, -1, 2)^T x$$

s.t.

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Proving Infeasibility

The following linear program is infeasible:

$$\max \quad (3,4,-1,2)^T x$$

s.t.

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Question

How can we prove this problem is, in fact, infeasible?

Proving Infeasibility

The following linear program is infeasible:

$$\begin{array}{ll} \max & (3, 4, -1, 2)^T x \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

Question

How can we prove this problem is, in fact, infeasible?

We cannot try all possible assignments of values to x_1, x_2, x_3, and x_4.

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrcrrrrlcr} & -1 \times (1): & (& -3 & 2 & 6 & -7 &)x & = & -6 \\ + & 2 \times (2): & (& 4 & -2 & -4 & 8 &)x & = & 4 \\ \hline & & (& 1 & 0 & 2 & 1 &)x & = & -2 \end{array} \qquad (\star)$$

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrrrrrrrl} & -1 \times (1): & (-3 & 2 & 6 & -7)x & = & -6 & \\ + & 2 \times (2): & (4 & -2 & -4 & 8)x & = & 4 & \\ \hline & & (1 & 0 & 2 & 1)x & = & -2 & (\star) \end{array}$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2).

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrcrrrrcr} & -1 \times (1): & (-3 & 2 & 6 & -7)x & = & -6 \\ + & 2 \times (2): & (4 & -2 & -4 & 8)x & = & 4 \\ \hline & & (1 & 0 & 2 & 1)x & = & -2 & \quad (\star) \end{array}$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrl} & -1 \times (1): & (-3 \quad 2 \quad 6 \quad -7)x = -6 \\ + & 2 \times (2): & (4 \quad -2 \quad -4 \quad 8)x = 4 \\ \hline & & (1 \quad 0 \quad 2 \quad 1)x = -2 \qquad (\star) \end{array}$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

$$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} \bar{x} = -2.$$

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrl} & -1 \times (1): & (-3 \quad 2 \quad 6 \quad -7)x = -6 \\ + & 2 \times (2): & (4 \quad -2 \quad -4 \quad 8)x = 4 \\ \hline & & (1 \quad 0 \quad 2 \quad 1)x = -2 \qquad (\star) \end{array}$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = -2.$$

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrcrrrrcr} & -1 \times (1): & (-3 & 2 & 6 & -7)x & = & -6 \\ + & 2 \times (2): & (4 & -2 & -4 & 8)x & = & 4 \\ \hline & & (1 & 0 & 2 & 1)x & = & -2 & \quad (\star) \end{array}$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} \bar{x}}_{\geq 0} = -2.$$

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrcrrrrcr} & -1 \times (1): & (-3 & 2 & 6 & -7)x & = & -6 \\ + & 2 \times (2): & (4 & -2 & -4 & 8)x & = & 4 \\ \hline & & (1 & 0 & 2 & 1)x & = & -2 \end{array} \qquad (\star)$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} \bar{x}}_{\geq 0} = \underbrace{-2}_{<0}.$$

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrl} & -1 \times (1): & (-3 \quad 2 \quad 6 \quad -7)x = -6 \\ + & 2 \times (2): & (4 \quad -2 \quad -4 \quad 8)x = 4 \\ \hline & & (1 \quad 0 \quad 2 \quad 1)x = -2 \qquad (\star) \end{array}$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} \bar{x}}_{\geq 0} = \underbrace{-2}_{<0}.$$

Contradiction!

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Construct a new equation:

$$(-1 \ \ 2) \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = (-1 \ \ 2) \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$(1 \ \ 0 \ \ 2 \ \ 1)x = -2 \qquad\qquad (\star)$$

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Construct a new equation:

$$(-1 \;\; 2) \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = (-1 \;\; 2) \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$(1 \;\; 0 \;\; 2 \;\; 1)x = -2 \qquad (\star)$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Construct a new equation:

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} x = -2 \qquad (\star)$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}.$$

Contradiction.

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b}$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$(1 \;\; 0 \;\; 2 \;\; 1) x = -2 \qquad\qquad (\star)$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}.$$

Contradiction.

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b} \qquad\qquad Ax = b$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$(1 \;\; 0 \;\; 2 \;\; 1)x = -2 \qquad\qquad (\star)$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}.$$

Contradiction.

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b} \qquad\qquad Ax = b$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$(1 \;\; 0 \;\; 2 \;\; 1)x = -2 \qquad (\star) \qquad\qquad y^T A x = y^T b$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}.$$

Contradiction.

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b} \qquad Ax = b$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} x = -2 \qquad (\star) \qquad y^T A x = y^T b$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}. \qquad \underbrace{y^T A}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{y^T b}_{<0}$$

Contradiction.

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b} \qquad Ax = b$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} x = -2 \qquad (\star) \qquad y^T A x = y^T b$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}. \qquad \underbrace{y^T A}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{y^T b}_{<0}$$

Contradiction.

This suggests the following result...

Proposition

There is no solution to $Ax = b, \ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Proposition

There is no solution to $Ax = b,\ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Exercise

Give a proof of this proposition.

Proposition

There is no solution to $Ax = b,\ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Exercise

Give a proof of this proposition.

Question

If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

Proposition

There is no solution to $Ax = b,\ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Exercise

Give a proof of this proposition.

Question

If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

YES!!!!!

Proposition

There is no solution to $Ax = b,\ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Exercise

Give a proof of this proposition.

Question

If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

YES!!!!!

Farkas' Lemma

If there is no solution to $Ax = b,\ x \geq \mathbb{0}$, then there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Proving Optimality

$$\max \quad z(x) := (-1 \ -4 \ \ 0 \ \ 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Proving Optimality

$$\max \quad z(x) := (-1\ -4\ \ 0\ \ 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Optimal solution:

$$\bar{x}_1 = 0$$

$$\bar{x}_2 = 0$$

$$\bar{x}_3 = 4$$

$$\bar{x}_4 = 5$$

Proving Optimality

$$\max \quad z(x) := (-1 \ -4 \ \ 0 \ \ 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
$$\bar{x}_3 = 4$$
$$\bar{x}_4 = 5$$

Question

How can we prove this solution is, in fact, optimal?

Proving Optimality

$$\max \quad z(x) := (-1\ -4\ \ 0\ \ 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
$$\bar{x}_3 = 4$$
$$\bar{x}_4 = 5$$

Question

How can we prove this solution is, in fact, optimal?

We cannot try all possible feasible solutions.

$$\max \quad z(x) := (-1 \; -4 \;\; 0 \;\; 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
$$\bar{x}_3 = 4$$
$$\bar{x}_4 = 5$$

Claim

- $\bar{x}$ is feasible solution of value 4.

$$
\begin{array}{ll}
\max & z(x) := (-1 - 4 \;\; 0 \;\; 0)x + 4 \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\
& x \geq \mathbb{0}
\end{array}
$$

Optimal solution:

$\bar{x}_1 = 0$

$\bar{x}_2 = 0$

$\bar{x}_3 = 4$

$\bar{x}_4 = 5$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)

$$\max \quad z(x) := (-1 - 4 \;\; 0 \;\; 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Optimal solution:

$\bar{x}_1 = 0$

$\bar{x}_2 = 0$

$\bar{x}_3 = 4$

$\bar{x}_4 = 5$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

$$\begin{array}{ll} \max & z(x) := (-1\ -4\ \ 0\ \ 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

Optimal solution:

$\bar{x}_1 = 0$

$\bar{x}_2 = 0$

$\bar{x}_3 = 4$

$\bar{x}_4 = 5$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

Proof

Let x' be an arbitrary feasible solution.

$$\begin{array}{ll} \max & z(x) := (-1\ -4\ \ 0\ \ 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

Optimal solution:

$\bar{x}_1 = 0$

$\bar{x}_2 = 0$

$\bar{x}_3 = 4$

$\bar{x}_4 = 5$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = (-1\ -4\ \ 0\ \ 0)x' + 4$$

$$\begin{aligned} \max \quad & z(x) := (-1\ -4\ \ 0\ \ 0)x + 4 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
$$\bar{x}_3 = 4$$
$$\bar{x}_4 = 5$$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1\ -4\ \ 0\ \ 0)}_{\leq \mathbb{0}} \underbrace{x'}_{\geq \mathbb{0}} + 4$$

$$\begin{aligned} \max \quad & z(x) := (-1 \; -4 \;\; 0 \;\; 0)x + 4 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
$$\bar{x}_3 = 4$$
$$\bar{x}_4 = 5$$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1 \; -4 \;\; 0 \;\; 0)x'}_{\leq 0} + 4$$

$$\begin{array}{ll} \max & z(x) := (-1 \; -4 \;\; 0 \;\; 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

Optimal solution:

$\bar{x}_1 = 0$

$\bar{x}_2 = 0$

$\bar{x}_3 = 4$

$\bar{x}_4 = 5$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1 \; -4 \;\; 0 \;\; 0)x'}_{\leq 0} + 4 \leq 4.$$

Proving Unboudedness

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Proving Unboudedness

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Problem
is unbounded

Question

How can we prove that this problem is unbounded?

Proving Unboudedness

$$\max \quad z := (-1\ \ 0\ \ 0\ \ 1)x$$

s.t.

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Problem
is unbounded

Question

How can we prove that this problem is unbounded?

Idea

Construct a family of feasible solutions $x(t)$ for all $t \geq 0$ and show that as t goes to infinity, the value of the objective function goes to infinity.

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

$$x(t) := \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \max \quad & z := (-1 \ \ 0 \ \ 0 \ \ 1)x \\ \text{s.t.} \quad & \\ & \begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

$$x(t) := \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

$$\begin{aligned} \max \quad & z := (-1 \ \ 0 \ \ 0 \ \ 1)x \\ \text{s.t.} \quad & \\ & \begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

$$x(t) := \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Claim 2

$z \to \infty$ when $t \to \infty$.

$$
\boxed{\begin{array}{ll}
\max & z := (-1\ \ 0\ \ 0\ \ 1)x \\
\text{s.t.} & \\
& \underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b} \\
& x \geq \mathbb{0}
\end{array}}
$$

$$
x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}
$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$\begin{aligned} \max \quad & z := (-1 \;\; 0 \;\; 0 \;\; 1)x \\ \text{s.t.} \quad & \underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b} \\ & x \geq \mathbb{0} \end{aligned}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$x(t) = \bar{x} + tr \geq \mathbb{0} \quad \text{for all} \quad t \geq 0$$

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$x(t) = \bar{x} + tr \geq \mathbb{0} \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad \bar{x}, r \geq \mathbb{0}.$$

$$\max \quad z := (-1 \ \ 0 \ \ 0 \ \ 1)x$$

s.t.

$$\underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$x(t) = \bar{x} + tr \geq \mathbb{0} \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad \bar{x}, r \geq \mathbb{0}.$$

$$Ax(t) =$$

$$\begin{aligned} \max \quad & z := (-1 \;\; 0 \;\; 0 \;\; 1)x \\ \text{s.t.} \quad & \underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b} \\ & x \geq \mathbb{0} \end{aligned}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$x(t) = \bar{x} + tr \geq \mathbb{0} \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad \bar{x}, r \geq \mathbb{0}.$$

$$Ax(t) = A\left[\bar{x} + tr\right] =$$

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$x(t) = \bar{x} + tr \geq \mathbb{0} \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad \bar{x}, r \geq \mathbb{0}.$$

$$Ax(t) = A\left[\bar{x} + tr\right] = \underbrace{A\bar{x}}_{b} + t \underbrace{Ar}_{\mathbb{0}} =$$

$$\max \quad z := \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} x$$

s.t.

$$\underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$x(t) = \bar{x} + tr \geq \mathbb{0} \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad \bar{x}, r \geq \mathbb{0}.$$

$$Ax(t) = A\left[\bar{x} + tr\right] = \underbrace{A\bar{x}}_{b} + t \underbrace{Ar}_{\mathbb{0}} = b.$$

$$\max \quad z := \underbrace{\begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}}_{c^T} x$$

s.t.

$$\underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 2

$z \to \infty$ when $t \to \infty$.

$$\max \quad z := \underbrace{\begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}}_{c^T} x$$

s.t.

$$\underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 2

$z \to \infty$ when $t \to \infty$.

Proof

$$z = c^T x(t)$$

$$\max \quad z := \underbrace{\begin{pmatrix}-1 & 0 & 0 & 1\end{pmatrix}}_{c^T} x$$

s.t.

$$\underbrace{\begin{pmatrix}-1 & -1 & 1 & 0\\ -2 & 1 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\ 1\end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix}0\\ 0\\ 2\\ 1\end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix}1\\ 0\\ 1\\ 2\end{pmatrix}}_{r}$$

Claim 2

$z \to \infty$ when $t \to \infty$.

Proof

$$z = c^T x(t) = c^T[\bar{x} + tr]$$

$$\max \quad z := \underbrace{\begin{pmatrix}-1 & 0 & 0 & 1\end{pmatrix}}_{c^T} x$$

s.t.

$$\underbrace{\begin{pmatrix}-1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2 \\ 1\end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix}0 \\ 0 \\ 2 \\ 1\end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix}1 \\ 0 \\ 1 \\ 2\end{pmatrix}}_{r}$$

Claim 2

$z \to \infty$ when $t \to \infty$.

Proof

$$z = c^T x(t) = c^T [\bar{x} + tr] = c^T \bar{x} + t \underbrace{c^T r}_{=1>0} .$$

Exercise

Generalize and prove the following proposition.

Exercise

Generalize and prove the following proposition.

Proposition

The linear program,

$$\max\{c^T x : Ax = b, x \geq \mathbb{0}\}$$

is unbounded if we can find $\bar{x}$ and r such that

$$\bar{x} \geq \mathbb{0}, \quad r \geq \mathbb{0}, \quad A\bar{x} = b, \quad Ar = \mathbb{0} \quad \text{and} \quad c^T r > 0.$$

Recap

1. For linear programs, exactly one of the following holds. It is
   (A) infeasible,
   (B) unbounded, or
   (C) has an optimal solution.

Recap

1. For linear programs, exactly one of the following holds. It is
   (A) infeasible,
   (B) unbounded, or
   (C) has an optimal solution.

2. If (A) occurs, there is a short proof of that fact.

Recap

1. For linear programs, exactly one of the following holds. It is
   (A) infeasible,
   (B) unbounded, or
   (C) has an optimal solution.

2. If (A) occurs, there is a short proof of that fact.

3. If (B) occurs, there is a short proof of that fact.

Recap

1. For linear programs, exactly one of the following holds. It is
   (A) infeasible,
   (B) unbounded, or
   (C) has an optimal solution.

2. If (A) occurs, there is a short proof of that fact.

3. If (B) occurs, there is a short proof of that fact.

4. For an optimal solution, there is a short proof that it is optimal.

Recap

1. For linear programs, exactly one of the following holds. It is
   (A) infeasible,
   (B) unbounded, or
   (C) has an optimal solution.

2. If (A) occurs, there is a short proof of that fact.

3. If (B) occurs, there is a short proof of that fact.

4. For an optimal solution, there is a short proof that it is optimal.

Remark

We have not yet shown you how to find such proofs.