Module 1: Formulations (LP Models)

Constrained Optimization

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where

- $n, m \in \mathbb{N}$,
- $b_1, \ldots, b_m \in \mathbb{R}$, and
- f, g_1, ..., g_m are functions with from $\mathbb{R}^n$ to $\mathbb{R}$.

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This class: all functions are affine.

Modeling: Linear Programs

Affine Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = a^T x + \beta$ for $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$.

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(ii) $f(x) = -3x_1 + 5x_3$ (linear)

(iii) $f(x) = 5x - 3\cos(x) + \sqrt{x}$ (not affine and not linear)

Definition

The optimization problem

$$\min\{f(x) : g_i(x) \leq b_i, \\ \forall 1 \leq i \leq m, x \in \mathbb{R}^n\} \tag{P}$$

is called a linear program if f is affine and $g_1, \ldots, g_m$ is finite number of linear functions.

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Instead of set notation, we often write LPs more verbosely

$$\begin{aligned} \max \quad & -2x_1 + 7x_3 \\ \text{subject to} \quad & x_1 + 7x_2 \leq 3 \\ & 3x_2 + 4x_3 \leq 2 \\ & x_1, x_3 \geq 0 \end{aligned}$$

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Second mathematical program is not an LP. Three reasons:

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Production revisited

$$
\begin{array}{lrcr}
\max & \multicolumn{3}{l}{300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u} \\
 & 11x_1 + 7x_2 + 6x_3 + 5x_4 & \leq & 700 \\
 & 4x_1 + 6x_2 + 5x_3 + 4x_4 & \leq & 500 \\
 & 8x_1 + 5x_2 + 5x_3 + 6x_4 & \leq & y_s \\
\text{s.t.} & 7x_1 + 8x_2 + 7x_3 + 4x_4 & \leq & y_u \\
 & y_s & \leq & 600 \\
 & y_u & \leq & 650 \\
 & x_1, x_2, x_3, x_4, y_u, y_s & \geq & 0.
\end{array}
$$

The mathematical program for WaterTech example from last class is in fact an LP!

Example: Multiperiod Models

Main feature of WaterTech production model:
Decisions about production levels have to be made once and for all.

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In practice, we often have to make a sequence of decision that influence each other.

One such example: Multiperiod Models:

- Time is split into periods,
- We have to make a decision in each period, and
- All decisions influence the final outcome.

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KW Oil is local supplier of heating oil

Needs to decide on how much oil to purchase in order to satisfy demand of its customers

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Demand (ℓ)	5000	8000	9000	6000

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Question: When should we purchase how much oil?

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Additional complication: Company has storage tank that

- Currently (beginning of month 1) contains 2000 litres of oil, and
- Has a capacity of 4000 litres of oil.

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Additional complication: Company has storage tank that

- Currently (beginning of month 1) contains 2000 litres of oil, and
- Has a capacity of 4000 litres of oil.

Assumption: Oil is delivered at beginning of month, and consumption occurs mid month

KW Oil Model – Variables

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

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$\longrightarrow$ variable t_i for $i \in [4]$

Objective function

Minimize cost of oil procurement.

Variables:

p_i : oil purchase in month i
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$$\min \quad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

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Objective function

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Constraints: when do

$$t_1, \ldots, t_4, p_1, \ldots, p_4$$

correspond to a feasible purchasing scheme?

Variables:

p_i : oil purchase in month i
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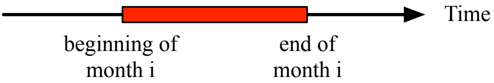
Constraints

Assumptions:



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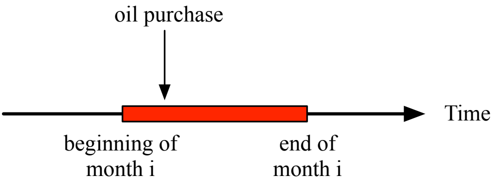

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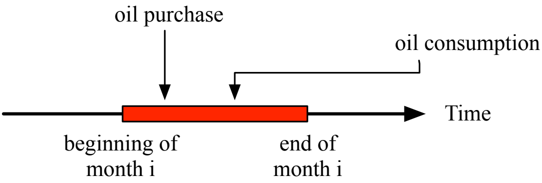

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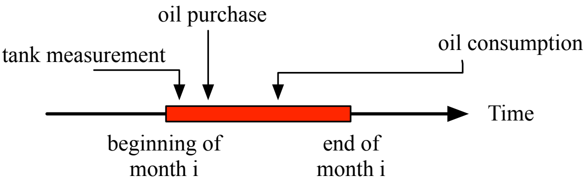

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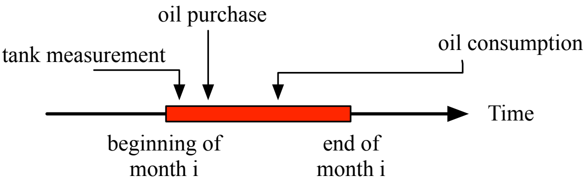

Variables:

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We need: $p_i + t_i \geq \{\text{demand in month } i\}$

Constraints

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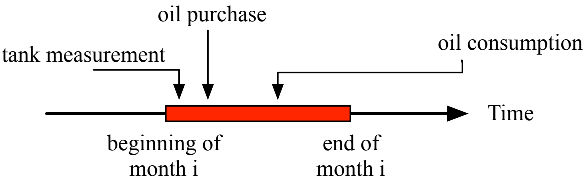

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[Balance Equation] $p_i + t_i = \{\text{demand in month } i\} + t_{i+1}$

Constraints

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Tank content in month 1: 2000 litres

Constraints

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Month 1:
$p_1 + 2000 = 5000 + t_2$

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

Tank content in month 1: 2000 litres

Constraints

[Balance Equation] $p_i + t_i =$ {demand in month i} $+t_{i+1}$

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Month 2:

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Month 3:

$p_3 + t_3 = 9000 + t_4$

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Tank content in month 1: 2000 litres

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$p_1 + 2000 = 5000 + t_2$

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Month 3:
$p_3 + t_3 = 9000 + t_4$

Month 4:
$p_4 + t_4 \geq 6000$

KW Oil: Entire LP

$$\min \quad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

subject to

$$\begin{aligned}
p_1 + t_1 &= 5000 + t_2 \\
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t_i &\leq 4000 \qquad (i = 2,3,4) \\
t_i, p_i &\geq 0 \qquad (i = 1,2,3,4)
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\end{aligned}$$

Solution: $p = (3000, 12000, 5000, 6000)^T$, and
$t = (2000, 0, 4000, 0)^T$

KW Oil: Add-Ons

Can easily capture additional features. E.g. ...

> Storage comes at a cost: storage cost is \$.15 per litre/month.

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Add $\sum_{i=1}^{4} .15t_i$ to objective.

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Minimize maximum #l of oil purchased over all months

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(i) We will need a new variable M for maximum #l purchased

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Can easily capture additional features. E.g. ...

Storage comes at a cost: storage cost is \$.15 per litre/month.

Add $\sum_{i=1}^{4} .15t_i$ to objective.

Minimize maximum #l of oil purchased over all months

(i) We will need a new variable M for maximum #l purchased

(ii) Will have to add constraints

$$
\begin{array}{llll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & \\
\text{subject to} & & & \\
& p_1 + t_1 & = & 5000 + t_2 \\
& p_2 + t_2 & = & 8000 + t_3 \\
& p_3 + t_3 & = & 9000 + t_4 \\
& p_4 + t_4 & \geq & 6000 \\
& t_1 & = & 2000 \\
& t_i & \leq & 4000 \quad (i = 2,3,4) \\
& t_i, p_i & \geq & 0 \quad (i = 1,2,3,4)
\end{array}
$$

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

$$
\begin{array}{lllll}
\min & 0.75p_1+0.72p_2+0.92p_3+0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1+t_1 & = & 5000+t_2 & \\
& p_2+t_2 & = & 8000+t_3 & \\
& p_3+t_3 & = & 9000+t_4 & \\
& p_4+t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i=2,3,4) \\
& t_i, p_i & \geq & 0 & (i=1,2,3,4)
\end{array}
$$

Goal: Minimize maximum #l of oil purchased over all months.

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

(ii) Add constraints

$$p_i \leq M$$

for all $i \in [4]$.

$$\begin{array}{lllll}
\text{min} & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2,3,4) \\
& t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array}$$

Goal: Minimize maximum #l of oil purchased over all months.

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

(ii) Add constraints

$$p_i \leq M$$

for all $i \in [4]$.

(iii) Done?

$$
\begin{array}{lllll}
\text{min} & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2, 3, 4) \\
& t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
$$

Goal: Minimize maximum #l of oil purchased over all months.

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

(ii) Add constraints

$$p_i \le M$$

for all $i \in [4]$.

(iii) Done? No! Need to replace objective function by

$$\min \quad M$$

$$\begin{array}{lll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & \\
\text{subject to} & & \\
& p_1 + t_1 = 5000 + t_2 & \\
& p_2 + t_2 = 8000 + t_3 & \\
& p_3 + t_3 = 9000 + t_4 & \\
& p_4 + t_4 \ge 6000 & \\
& t_1 = 2000 & \\
& t_i \le 4000 & (i = 2,3,4) \\
& t_i, p_i \ge 0 & (i = 1,2,3,4)
\end{array}$$

Goal: Minimize maximum #l of oil purchased over all months.

Minimizing the Maximum Purchase: LP

$$\min \quad M$$

s.t.

$$\begin{array}{lcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
p_i & \leq & M & (i = 1, 2, 3, 4) \\
t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}$$

KW Oil: Correctness

Why is this a correct model?

$$\min \quad M$$

s.t.

$$\begin{aligned}
p_1 + t_1 &= 5000 + t_2 \\
p_2 + t_2 &= 8000 + t_3 \\
p_3 + t_3 &= 9000 + t_4 \\
p_4 + t_4 &\geq 6000 \\
t_1 &= 2000 \\
t_i &\leq 4000 && (i = 2, 3, 4) \\
p_i &\leq M && (i = 1, 2, 3, 4) \\
t_i, p_i &\geq 0 && (i = 1, 2, 3, 4)
\end{aligned}$$

KW Oil: Correctness

Why is this a correct model?

Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

$$
\begin{array}{lllll}
\min & M & & & \\
\text{s.t.} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2, 3, 4) \\
& p_i & \leq & M & (i = 1, 2, 3, 4) \\
& t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
$$

KW Oil: Correctness

Why is this a correct model?

Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

Clearly: $M \geq \max_i p_i$

$$
\begin{array}{lllll}
\min & M & & & \\
\text{s.t.} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
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& t_i & \leq & 4000 & (i = 2, 3, 4) \\
& p_i & \leq & M & (i = 1, 2, 3, 4) \\
& t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
$$

KW Oil: Correctness

Why is this a correct model?

Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

Clearly: $M \geq \max_i p_i$

Since M, p, t is optimal we must have $M = \max_i p_i$.
Why?

$$\min \quad M$$

s.t.

$$\begin{array}{rcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
p_i & \leq & M & (i = 1, 2, 3, 4) \\
t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}$$