Module 1: Formulations (Shortest Paths)
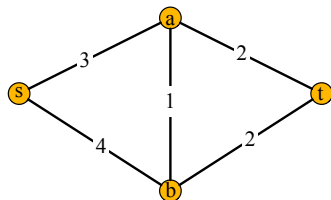
Recap: **Shortest Paths**
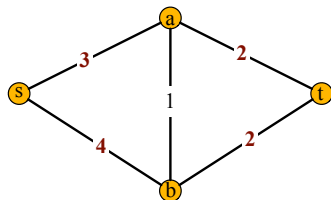
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- Graph $G = (V, E)$

Recap: **Shortest Paths**

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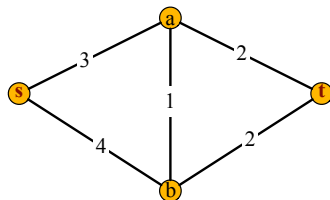
- Graph $G = (V, E)$

- Non-negative edge lengths c_e for all $e \in E$

Recap: **Shortest Paths**

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- Vertices $s, t \in V$

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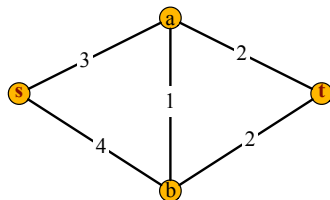Goal: Compute an s, t-path of smallest total length.

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- Graph $G = (V, E)$

- Non-negative edge lengths c_e for all $e \in E$

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Recall: P is an s, t-path if it is of the form

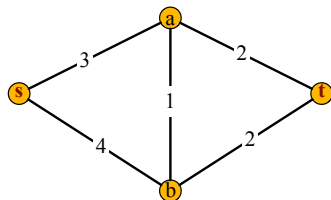$$v_1 v_2, \ v_2 v_3, \ \ldots, \ v_{k-1} v_k$$

and

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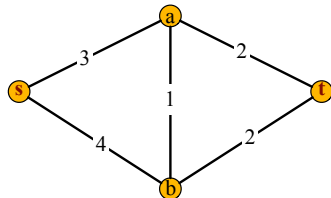1. $v_iv_{i+1} \in E$ for all $i \in \{1, \ldots, k-1\}$,

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$$v_1 v_2, \; v_2 v_3, \; \ldots, \; v_{k-1} v_k$$

and

1. $v_i v_{i+1} \in E$ for all $i \in \{1, \ldots, k-1\}$,

2. $v_i \neq v_j$ for all $i \neq j$, and
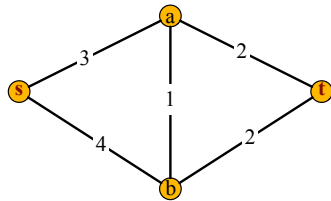
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1. $v_i v_{i+1} \in E$ for all $i \in \{1, \ldots, k-1\}$,

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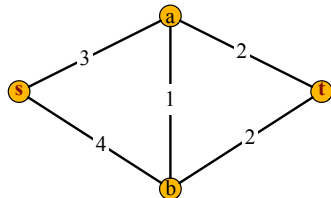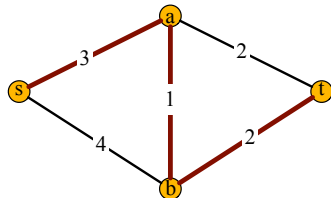3. $v_1 = s$ and $v_k = t$.

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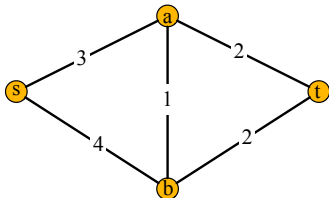$$v_1 v_2, \; v_2 v_3, \; \ldots, \; v_{k-1} v_k$$

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3. $v_1 = s$ and $v_k = t$.

E.g., $P = sa, \, ab, \, bt$

Shortest Path Problem: Given
$G = (V, E)$, $c_e \geq 0$ for all $e \in E$,
and $s, t \in V$, compute an s, t-path
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Shortest Path Problem: Given $G = (V, E)$, $c_e \geq 0$ for all $e \in E$, and $s, t \in V$, compute an s, t-path of smallest total length.

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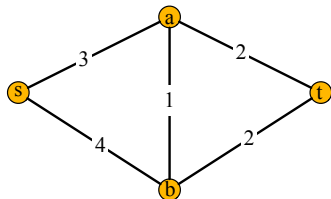Useful Observation: Let $C \subseteq E$ be a set of edges whose removal disconnects s and t.

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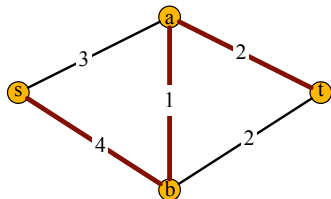
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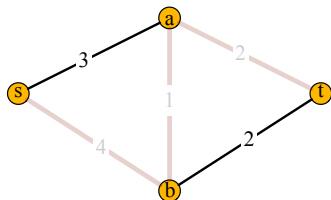Useful Observation: Let $C \subseteq E$ be a set of edges whose removal disconnects s and t.

$\longrightarrow$ Every s, t-path P must have at least
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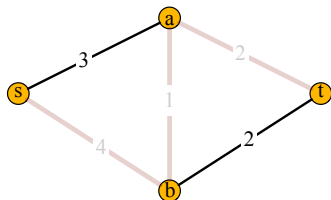Useful Observation: Let $C \subseteq E$ be a set of edges whose removal disconnects s and t.

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Definition

For $S \subseteq V$, we let $\delta(S)$ be the set of edges with exactly one endpoint in S.
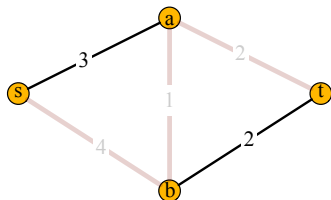
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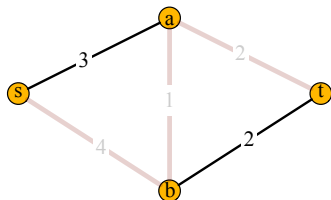For $S \subseteq V$, we let $\delta(S)$ be the set of edges with exactly one endpoint in S.

$$\delta(S) = \{uv \in E : u \in S, v \notin S\}$$

Cuts

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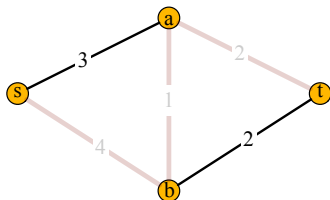1. $S = \{s\} \to \delta(S) = \{sa, sb\}$

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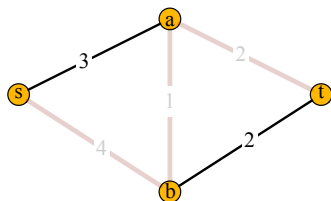

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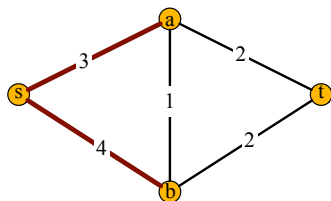

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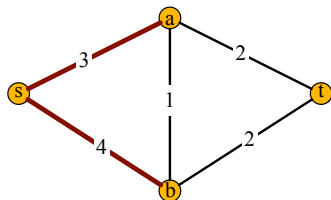
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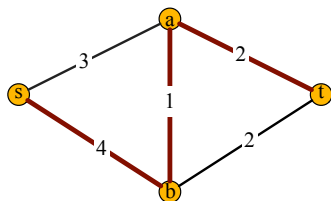

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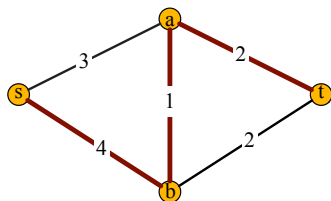2. $S = \{s, a\} \rightarrow \delta(S) = \{ab, at, sb\}$

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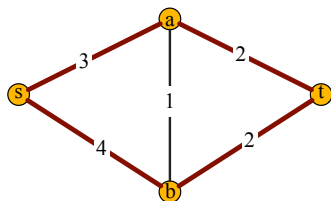3. $S = \{a, b\} \rightarrow \delta(S) = \{sa, sb, at, bt\}$
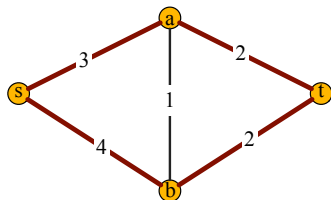

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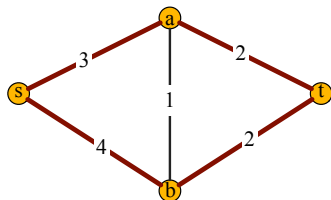E.g., 1 and 2 are s, t-cuts, 3 is not.

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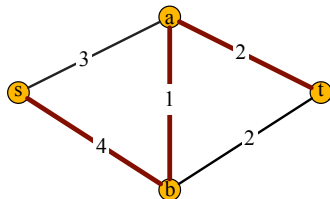
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E.g., $\delta(\{s, a\}) = \{sb, ab, at\}$ is an s, t-cut.

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If P is an s, t-path and $\delta(S)$ is an s, t-cut, then P must have an edge from $\delta(S)$.
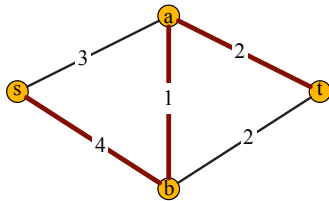
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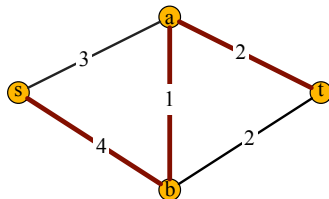
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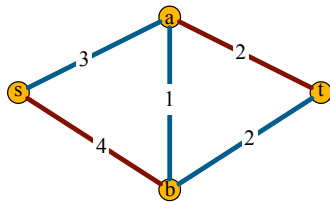
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Remark

If $S \subseteq E$ contains at least one edge from every s, t-cut, then S contains an s, t-path.
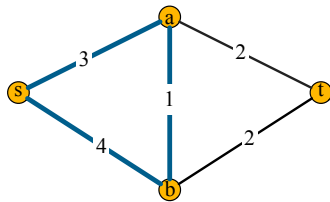
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Proof: (by contradiction)

- Suppose S has an edge from every s, t-cut, but S has no s, t-path.
- Let R be the set of vertices reachable from s in S:

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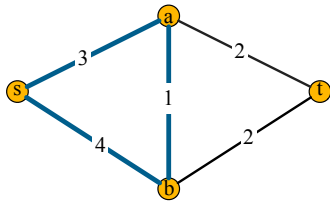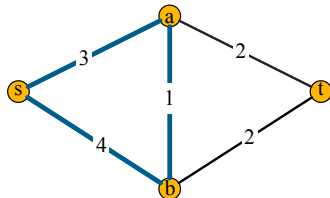- $\delta(R)$ is an s, t-cut since $s \in R$ and $t \notin R$.

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- $\delta(R)$ is an s, t-cut since $s \in R$ and $t \notin R$.

- Note: There cannot be an edge $uv \in S$ with $u \in R$ and $v \notin R$.

Cuts

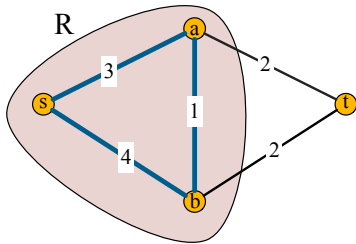## Remark

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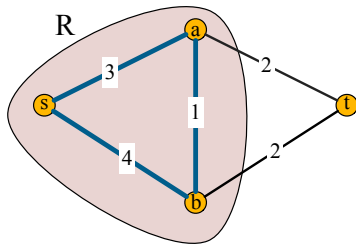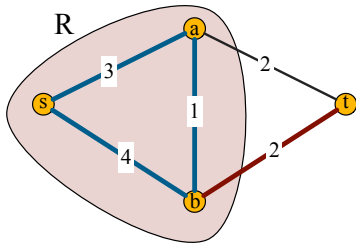
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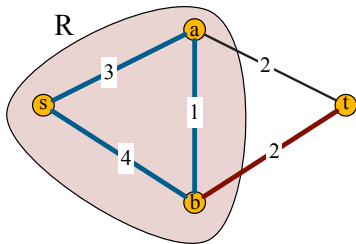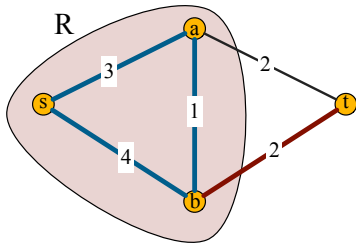- Note: There cannot be an edge $uv \in S$ with $u \in R$ and $v \notin R$. Otherwise: v should have been in R!

$\longrightarrow \delta(R) \cap S = \emptyset$.
  Contradiction!

An IP for Shortest Paths
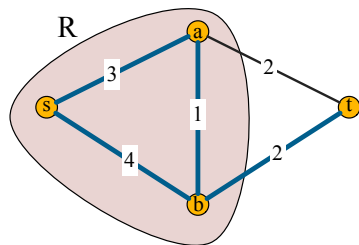
Variables: We have one binary variable x_e for each edge $e \in E$.

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If $S \subseteq E$ contains at least one edge from every s, t-cut, then S contains an s, t-path.

An IP for Shortest Paths

Variables: We have one binary variable x_e for each edge $e \in E$. We want:

$$x_e = \begin{cases} 1 : e \in P \\ 0 : \text{otherwise} \end{cases}$$

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Constraints: We have one constraint for each s, t-cut $\delta(U)$, forcing P to have an edge from $\delta(S)$.

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$$\sum (x_e \, : \, e \in \delta(U)) \geq 1 \qquad (1)$$
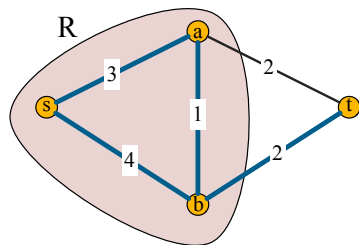
for all s, t-cuts $\delta(U)$.

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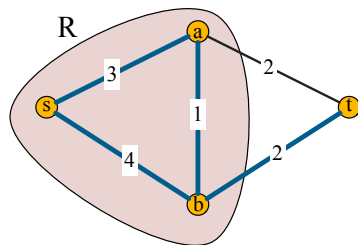$$x_e = \begin{cases} 1 : e \in P \\ 0 : \text{otherwise} \end{cases}$$

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Objective: $\sum (c_e x_e \: : \: e \in E)$

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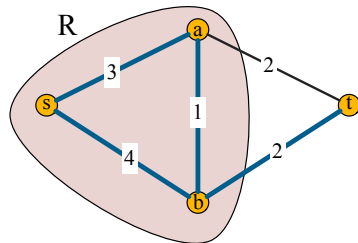
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$$\text{s.t.} \quad \begin{array}{c} \\ \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} \begin{array}{ccccc} sa & sb & ab & at & bt \\ \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \end{array} x \geq \mathbb{1}$$
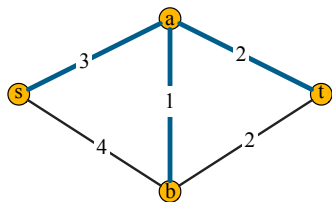
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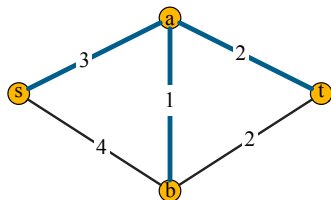Then let $x_e = 1$. This is cheaper and maintains feasibility!

For a binary solution x, define

$$S_x = \{e \in E \,:\, x_e = 1\}.$$

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Note: If x is feasible for an IP, then S_x satisfies the remark, but S_x may contain more than just an s, t-path!

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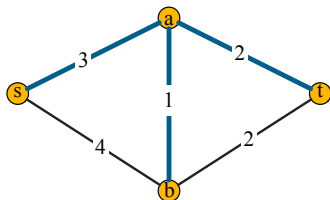
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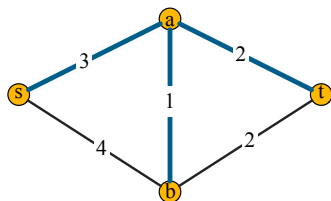E.g., $x_e = 1$ for all blue edges in the figure and $x_e = 0$ otherwise. Then,

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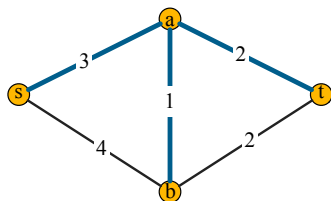
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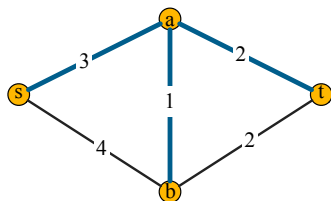
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Remark

If x is an optimal solution for the above IP and $c_e > 0$ for all $e \in E$, then S_x contains the edges of a shortest s, t-path.

Recap

- Given $G = (V, E)$ and $U \subseteq V$, we define

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- If $S \subseteq E$ intersects every s, t-cut $\delta(U)$, then S contains an s, t-path.

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