Module 2: Linear Programs (Basis)

Notation

Notation

Consider

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

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$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.

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Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

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$B = \{1, 2, 3\}$

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$$A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notation

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\end{array}
$$

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Notation

Let B be a subset of column indices.
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$$B = \{1, 3, 4\}$$

Notation

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Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{1, 3, 4\}$$

$$A_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Notation

Consider

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{5\}$$

Notation

Consider

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\end{array}
$$

Notation

Let B be a subset of column indices.
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$$B = \{5\}$$

$$A_{\{5\}} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

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Notation

A_j denotes column j of A.

Notation

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\end{array}
$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{5\} \qquad A_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Notation

A_j denotes column j of A.

Basis

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Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

Basis

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Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

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Let B be a subset of column indices. B is a basis if

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Is $B = \{1, 2, 3\}$ a basis?

Basis

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Is $B = \{1, 5\}$ a basis?

$$A_B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

NO

A_B is not square

Basis

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$$\begin{array}{c} \begin{matrix} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{matrix} \\ A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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Is $B = \{2, 3, 4\}$ a basis?

Basis

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$$
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$$A_B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Basis

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Is $B = \{2, 3, 4\}$ a basis?

$$A_B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

NO

columns of A_B are dependent

Question

Does every matrix have a basis?

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Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Theorem

Max number of independent columns $=$
Max number of independent rows.

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Theorem

Max number of independent columns $=$
Max number of independent rows.

Remark

Let A be a matrix with independent rows. Then B is a basis if and only if B is a maximal set of independent columns of A.

Basic Solutions

Basic Solutions

$$\underbrace{\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}
\end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

Basic Solutions

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Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,

Basic Solutions

$$\underbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \end{array}}_{} $$

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Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Example

Basis $B = \{1, 2, 4\}$.

Basic Solutions

$$\underbrace{\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \end{array}}_{}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
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Example

Basis $B = \{1, 2, 4\}$. Then

- x_1, x_2, x_4 are the basic variables,

Basic Solutions

$$\underbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \multicolumn{5}{c}{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Example

Basis $B = \{1, 2, 4\}$. Then

- x_1, x_2, x_4 are the basic variables, and
- x_3, x_5 are the non-basic variables.

Basic Solutions

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Basic Solutions

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Definition

x is a basic solution for basis B if

Basic Solutions

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Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

Basic Solutions

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Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$: basic sol'n for $B = \{1, 2, 3\}$

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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(1) $Ax = b$

Basic Solutions

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$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}: \text{ basic sol'n for } B = \{1, 2, 3\}$$

(1) $Ax = b$ ✓

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Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}: \text{ basic sol'n for } B = \{1, 2, 3\}$$

(1) $Ax = b$ ✓

(2) $x_4 = x_5 = 0$

Basic Solutions

$$\underbrace{\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \end{array}}_{}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$: basic sol'n for $B = \{1, 2, 3\}$

(1) $Ax = b$ ✓

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Problem

Find a basic solution x for the basis $B = \{1, 4\}$?

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Problem

Find a basic solution x for the basis $B = \{1, 4\}$?

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} x$$

$$\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{b}$$

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$$\Rightarrow \qquad \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

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Thus, the basic solution is $x = (4,\ 0,\ 0,\ 2)^\top$.

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Question

Did we have a choice for a basic solution x given $B = \{1, 4\}$?

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Problem

Find a basic solution x for the basis $B = \{1, 4\}$?

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} x$$
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Thus, the basic solution is $x = (4,\ 0,\ 0,\ 2)^\top$.

Question

Did we have a choice for a basic solution x given $B = \{1, 4\}$? NO!

Basic Solutions – Uniqueness

Proposition

Consider $Ax = b$ and a basis B of A.
Then there exists a unique basic solution x for B.

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For $B = \{1, 2, 4\}$,

$$A_B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \quad \text{(basic variables)}$$

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columns of A_B and elements of x_B are ordered by B!

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$$b = Ax = \sum_j A_j x_j$$

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$$\begin{aligned} b =& Ax = \sum_j A_j x_j \\ =& \sum_{j \in B} A_j x_j + \sum_{j \notin B} A_j \underbrace{x_j}_{=0} \end{aligned}$$

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b =& Ax = \sum_j A_j x_j \\
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Since B is a basis, it implies A_B is non-singular, i.e., A_B^{-1} exists.

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$$

Since B is a basis, it implies A_B is non-singular, i.e., A_B^{-1} exists.

Hence, $x_B = A_B^{-1} b$.

When Is a Vector Basic?

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Question

Is $x = (0, 0, 3, 0, 3)^\top$ basic?

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Is $x = (0, 1, 0, 1, 0)^\top$ basic?

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By contradiction. Suppose x is basic for basis B.

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Multiple Bases for a Basic Solution

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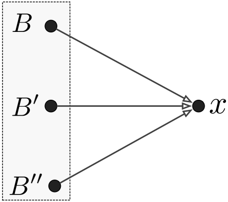

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Problem in SEF:

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Definition

A basic solution x of $Ax = b$ is feasible if $x \geq 0$, i.e., if it is feasible for (P).

Consider the system

$$Ax = b$$

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(6) A basic solution is feasible if it is non-negative.