Module 5: Integer Programs (IP versus LP)

LP versus IP

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Remark

We cannot PROVE an algorithm that is guaranteed to be fast does not exist, but we can show that it is "highly unlikely".

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Let us look at an example...

A Bad Example

A Bad Example

Proposition

The following IP,

$$
\boxed{
\begin{array}{l}
\max \quad x_1 - \sqrt{2}x_2 \\
\text{s.t.} \\
\qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
\qquad x_1, x_2 \text{ integer}
\end{array}
}
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is feasible, bounded, and has no optimal solution.

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- It has no optimal solution. ???

$$\max \quad x_1 - \sqrt{2}x_2$$

s.t.

$$\begin{aligned} x_1 &\leq \sqrt{2}x_2 \\ x_1, x_2 &\geq 1 \end{aligned}$$

x_1, x_2 integer

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Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

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⇒

contradiction !!!

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$$\begin{aligned} x_1' &\stackrel{?}{\leq} \sqrt{2}x_2' \\ 2x_1 + 2x_2 &\stackrel{?}{\leq} \sqrt{2}\,(x_1 + 2x_2) \end{aligned} \qquad \Longleftrightarrow$$

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$$(2x_1 + 2x_2) - \sqrt{2}(x_1 + 2x_2)\sqrt{2} \overset{?}{>} x_1 - \sqrt{2}x_2$$

$$
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Simplifying (takes a little work), we obtain

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- $\geq$ since x_1, x_2 are feasible for (IP)

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Simplifying (takes a little work), we obtain

$$\sqrt{2}x_2 \overset{?}{>} x_1$$

- $\geq$ since x_1, x_2 are feasible for (IP)
- $>$ otherwise $\sqrt{2} = \frac{x_1}{x_2}$ but $\sqrt{2}$ is not a rational number

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Good News:

- IPs can formulate a huge number of practical problems.
- Commercial codes solve many of these problems routinely.
- Some of the theory developed for LPs extends to IPs.

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- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.
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Integer Programming can, in principle, be reduced to Linear Programming.

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This will NOT give us a practical procedure to solve IPs,

but it will suggest a strategy.

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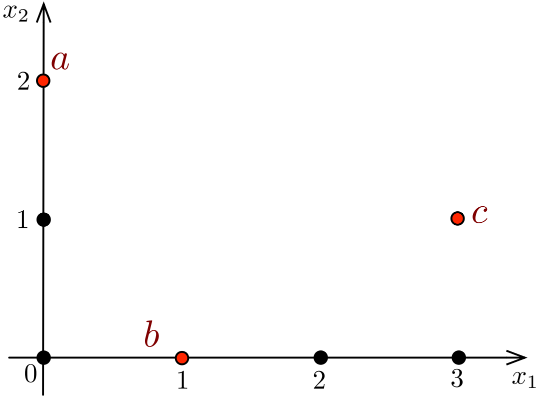

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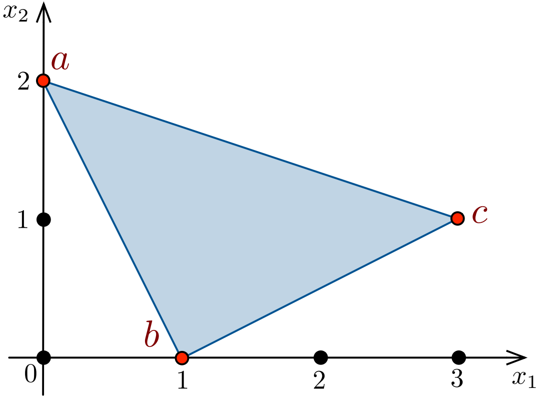

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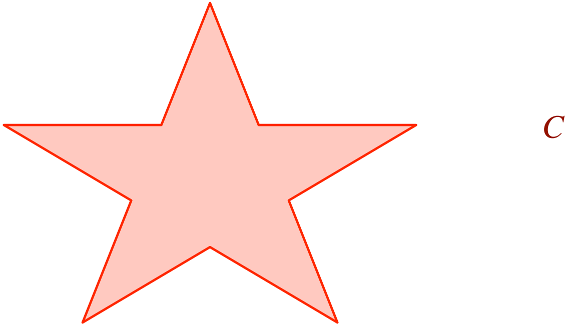

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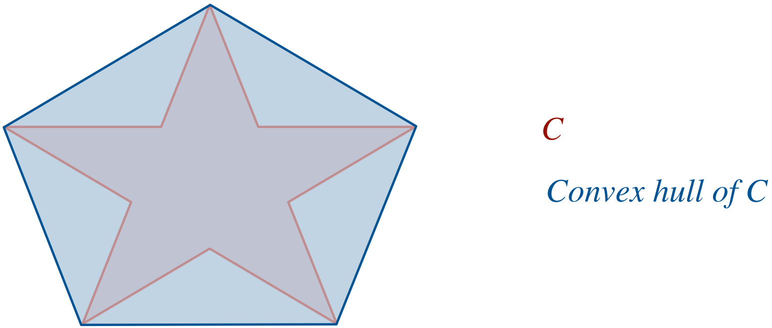

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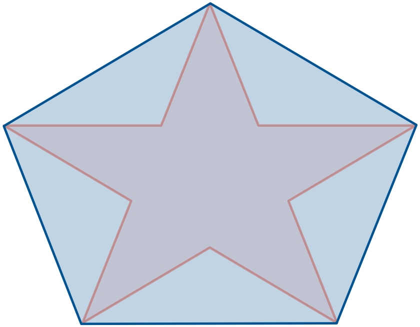

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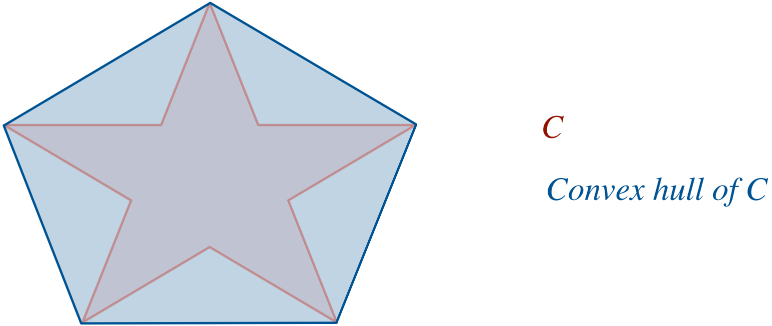

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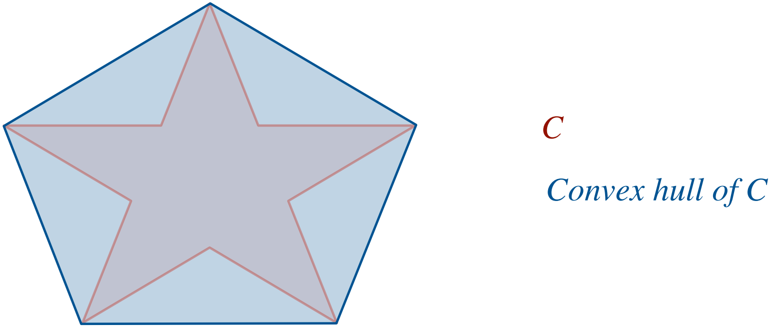

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➡ The notion of a convex hull is well defined.

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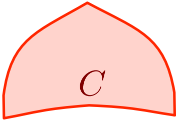

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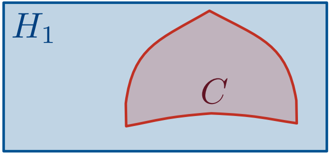

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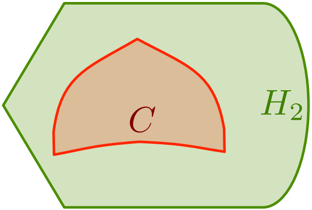

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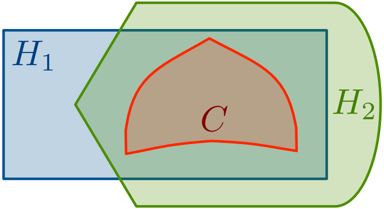

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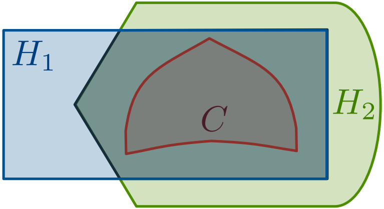

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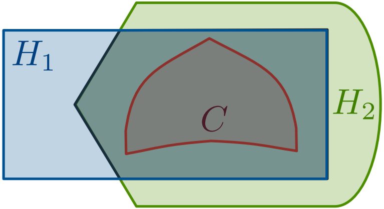

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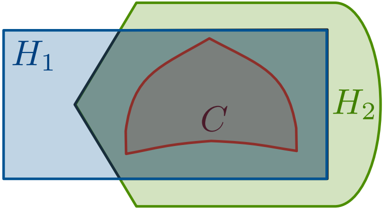

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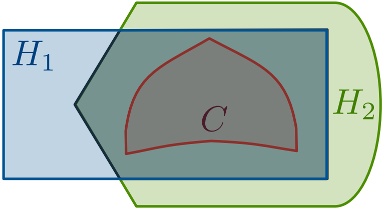

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However, $H_1 \cap H_2$ is smaller than both H_1 and H_2. This is a contradiction.

Convex Hulls and Integer Programs

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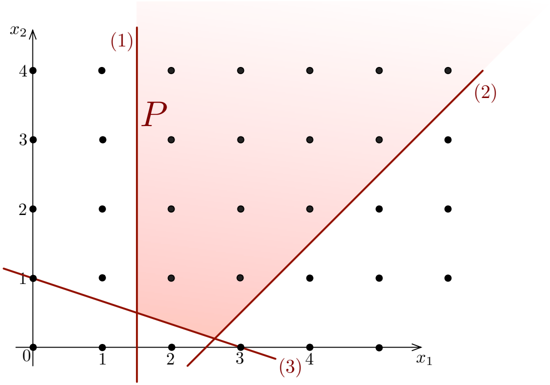

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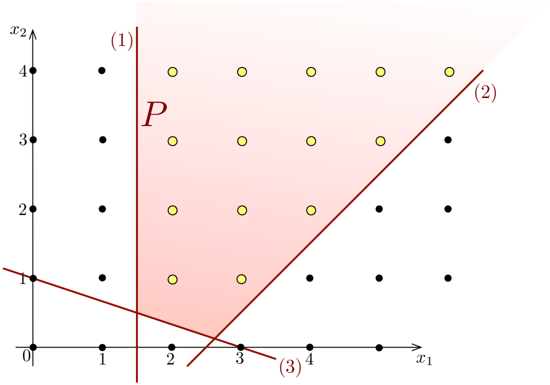

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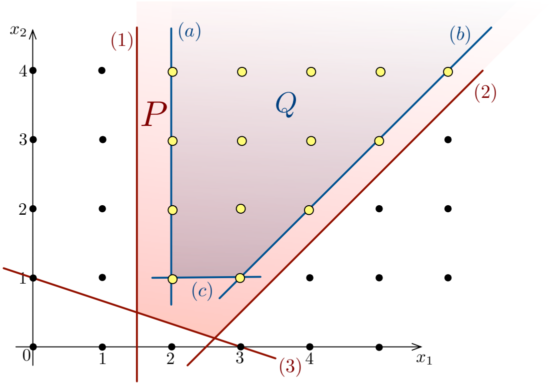

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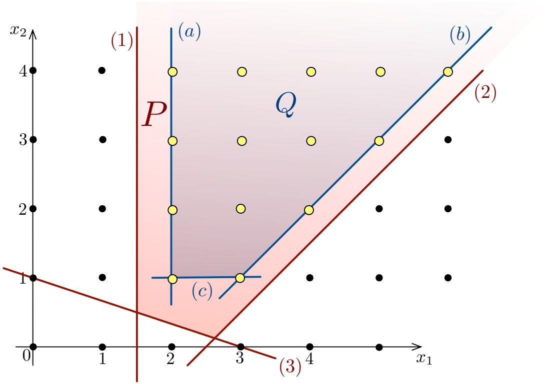

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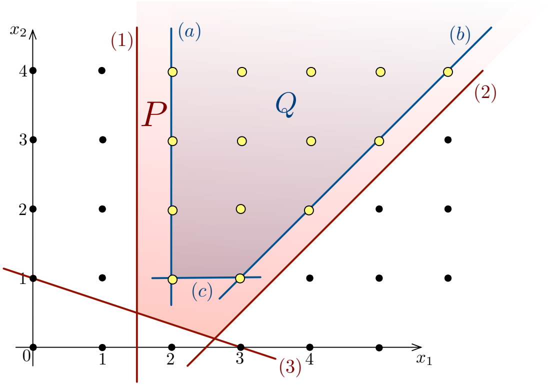

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POLYHEDRON

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Goal: Use Meyer's theorem to reduce the problem of solving *Integer Programs*, to the problem of solving *Linear Program*.

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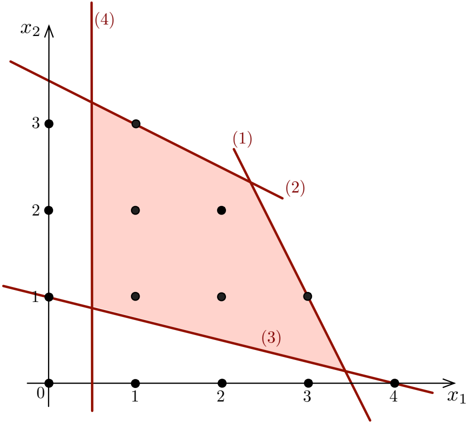

Feasible region of the LP relaxation of (IP)

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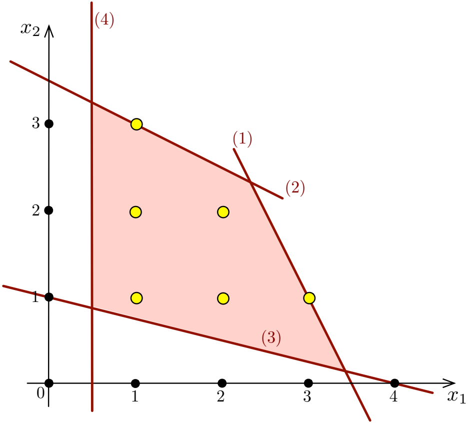

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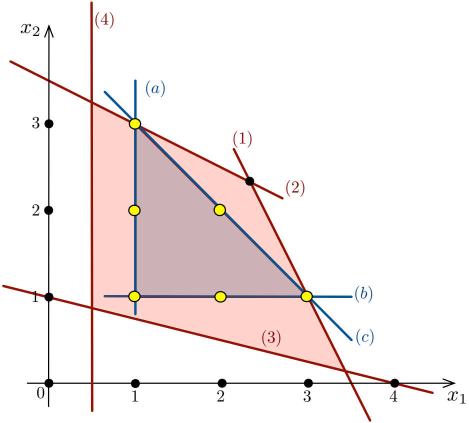
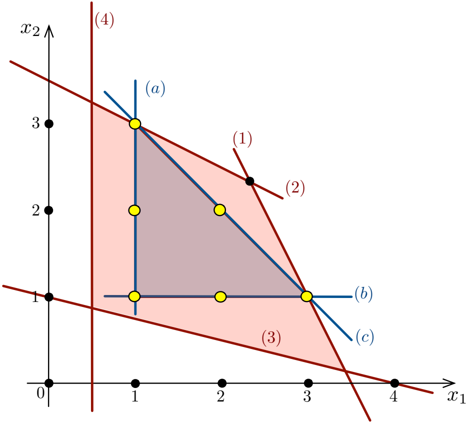


The convex hull of the feasible region of (IP)

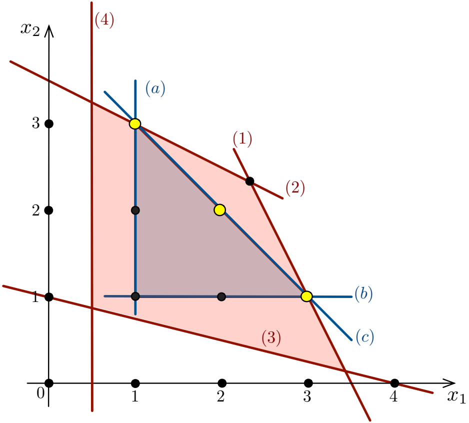
$$\max \left\{ \begin{pmatrix} 1 & 1 \end{pmatrix} x : \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -4 \\ -1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 7 \\ 7 \\ -4 \\ -\frac{1}{2} \end{pmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix} \quad x \text{ integer} \right\} \qquad \text{(IP)}$$



The convex hull of the feasible region of (IP)

$$\max \left\{ \begin{pmatrix} 1 & 1 \end{pmatrix} x : \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} \begin{matrix} (a) \\ (b) \\ (c) \end{matrix} \right\} \qquad \text{(LP)}$$

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}2 & 1\\ 1 & 2\\ -1 & -4\\ -1 & 0\end{pmatrix}x \le \begin{pmatrix}7\\ 7\\ -4\\ -\frac{1}{2}\end{pmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix} \quad x \text{ integer}\right\} \qquad \text{(IP)}$$



Optimal solutions of (IP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix}x \le \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix} \quad \begin{matrix}(a)\\(b)\\(c)\end{matrix}\right\} \qquad \text{(LP)}$$

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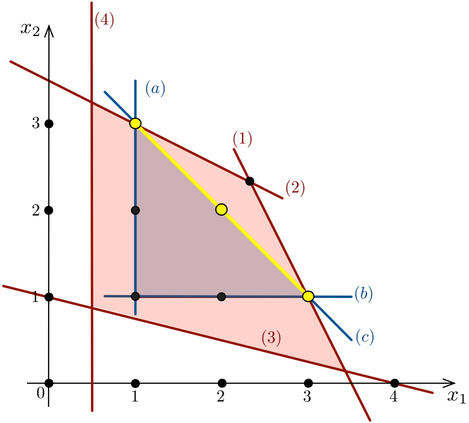

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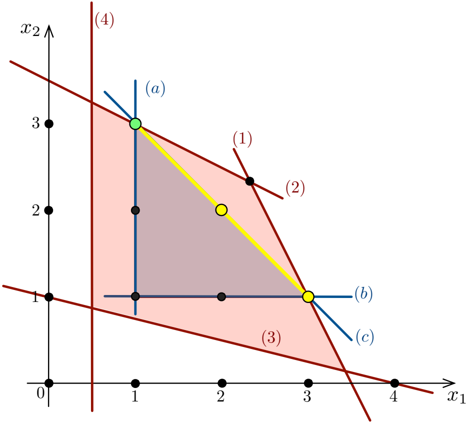

Extreme optimal solution of (LP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix}x \le \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix} \begin{matrix}(a)\\(b)\\(c)\end{matrix}\right\} \qquad \text{(LP)}$$

$$\max\{c^\top x : Ax \le b, x \text{ integer}\} \tag{IP}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

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This is NOT a practical way to solve an Integer Program.

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Step 2. Use Simplex to find an extreme optimal solution.

Remark

This is NOT a practical way to solve an Integer Program.

WHY NOT?

- We do not know how to compute A', b', and
- A', b' can be MUCH more complicated than A, b.

Question

How do we fix these problems?

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Idea

Construct an approximation of the convex hull of the solutions of (IP).

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Recap

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- Integer Programs are much harder to solve than Linear Programs.

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- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.

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- Integer Programs are much harder to solve than Linear Programs.
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- We defined the notion of convex hulls.

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Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.
- We defined the notion of convex hulls.
- The convex hull of the integer points in a rational polyhedron is a polyhedron.

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- Integer Programs are much harder to solve than Linear Programs.
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Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.
- We defined the notion of convex hulls.
- The convex hull of the integer points in a rational polyhedron is a polyhedron.
- Integer programming reduces to Linear programming, but it is NOT a practical reduction.