Optimization - assignment 1

Task 1.1:

Let $x_1, ..., x_5 \in \mathbb{R}$ denote the amount of savings for each food. x_1 thereby stands for the amount of servings of raw carrots, x_2 for the amount of savings of baked potatoes and so on.
The underlying LP then reads:

$$\begin{aligned} \min\ & 0.14x_1 + 0.12x_2 + 0.2x_3 + 0.75x_4 + 0.15x_5 \\ \text{s.t. } & 23x_1 + 171x_2 + 65x_3 + 112x_4 + 188x_5 \geq 2000 \\ & 0.1x_1 + 0.2x_2 + 9.3x_4 + 16x_5 \geq 50 \\ & 0.6x_1 + 3.7x_2 + 2.2x_3 + 7x_4 + 7.7x_5 \geq 100 \\ & 6x_1 + 30x_2 + 13x_3 + 2x_5 \geq 250 \\ & x \geq 0 \end{aligned}$$

With $x = (x_1, ..., x_5)^T$.

Task 1.2:

(a) *Proof by contradiction.* Let (x, y) be the solution vector of the given optimization problem. Assume now that there is a vector $\tilde{x} \in \mathbb{R}^n$ with smaller deviation than x, i.e.

$$\sum_{i=1}^{n} |b_i - \sum_{j=1}^{n} a_{ij}\tilde{x}_j| < \sum_{i=1}^{n} |b_i - \sum_{j=1}^{n} a_{ij}x_j| \,.$$

Then,

$$\sum_{i=1}^{m} y_i \geq \sum_{i=1}^{m} |b_i - \sum_{j=1}^{n} a_{ij}x_j| > \sum_{i=1}^{m} |b_i - \sum_{j=1}^{n} a_{ij}\tilde{x}_j|$$

which contradicts the assumption that (x, y) is the optimal solution to the LP since $(\tilde{x}, \tilde{y})$ with $\tilde{y}_i = |b_i - \sum_{j=1}^{n} a_{ij}\tilde{x}_j| \ \forall i \in [m]$ is a feasible solution with a smaller objective function evaluation.

Thus, the assumption is wrong and x is a vector of minimum deviation. □

(b) Due to the absolute values on the left hand side of the constraints, the constraint function is not linear which makes this optimization problem to a nonlinear one. To overcome this issue, one can formulate:

$$\begin{aligned} b_i - \sum_{j=1}^{n} a_{ij}x_j &\leq y_i \\ \text{and} \quad -b_i + \sum_{j=1}^{n} a_{ij}x_j &\leq y_i \quad \forall i \in [m] \end{aligned}$$

instead of $y_i \geq |b_i - \sum_{j=1}^{n} a_{ij}x_j| \ \forall i \in [m]$ since $|a| \leq b \iff a \leq b \wedge -a \leq b$ for general $a, b \in \mathbb{R}$

(c) The corresponding LP can be formulated as:

$$\min y \quad \text{subject to} \quad |b_i - \sum_{j=1}^{n} a_{ij}x_j| \leq y \quad \forall i \in [m]$$

analogously to (b), this can also be formulated as

$$\begin{aligned} &\min y \\ &\text{s.t. } b_i - \sum_{j=1}^{n} a_{ij}x_j \leq y \quad \forall i \in [m] \\ &\qquad \sum_{j=1}^{n} a_{ij}x_j - b_i \leq y \quad \forall i \in [m] \end{aligned}$$

which transforms the formulated optimization problem to a LP.