Introduction to Optimization

Part 1: Formulations (Overview)

Outline

Optimization - Abstract Perspective

- Abstractly, we will focus on problems of the following form:
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- Very general problem that is enormously useful in virtually very branch of industry.
- Bad news: the above problem is notoriously hard to solve (and may not even be well-defined).

Optimization - Important Special Cases

- *Abstract optimization problem (P):*
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  - (C) Nonlinear Programming. A is given by *non-linear* constraints, and f is a *non-linear* function.

Optimization - Typical Workflow

Typical development process has three stages.

- Starting point is English language description of practical problem

Practical Problem:
- Description in plain language
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Practical Implementation:
- E.g., OPL (IBM ILOG)

Optimization - Typical Workflow

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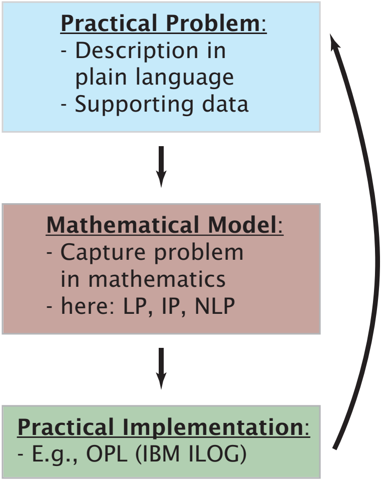
- Finally, feed model and data into a solver.
- Iterate!



Optimization in Practice

Optimization is everywhere! Some examples:

- Booking hotel rooms or airline tickets,
- Setting the market price of a kwh of electricity,
- Determining an “optimal” portfolio of stocks,
- Computing energy efficient circuits in chip design,
- and many more!

CSX Rail

- One of the largest transport suppliers in the United States.
- CSX operates 21000 miles of rail network
- 11 Billion in annual revenue
- Serves 23 states, Ontario and Quebec
- Operates 1200 trains per day

- Has a fleet of 3800 locomotives, and more than 100000 freight cars
- Transports 7.4 million car loads per year

Optimization @ CSX Rail

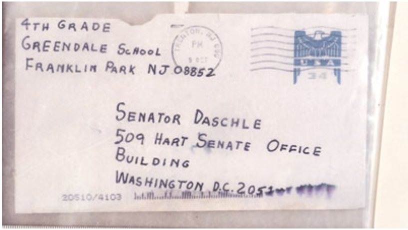
- [Acharya, Sellers, Gorman '10] use mathematical programming to optimally allocate and reposition empty railcars dynamically

  - Implementing system yields the following estimated benefits for CSX:

    Annual savings: $51 million
    Avoided rail car capital investment: $1.4 billion

Optimization in Disease Control

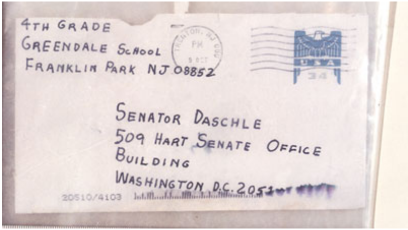
- [Lee et al. '13] Use mathematical programming to prepare for disease outbreak and medical catastrophes.
- Where should we place medical dispensing facilities, and how should we staff these in order to disseminate medication as quickly as possible to population?
- How should dispensing be scheduled?



2001 Anthrax letter sent to Senator T. Daschle

Optimization in Disease Control

- In collaboration with the Center for Disease Control, [Lee et al. '13] develop decision support suite RealOpt
- Suite is being used by $\approx$ 6500 public health and emergency directors in the USA to design, place and staff medical dispensing centers
- In tests, throughput in medical dispensing centers increases by several orders of magnitude.



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Product	Machine 1	Machine 2	Skilled Labor	Unskilled Labor	Unit Sale Price
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E.g.: producing a unit of product 3 requires 6h on machine 1, 5h on machine 2, 5h of skilled, and 7h of unskilled labour. It can be sold at $220 per unit.

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Restrictions:

- WaterTech has available 700h of machine 1, and 500h of machine 2 time
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Formulate this as a mathematical program!

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t	Machine 1	Machine 2	Skil
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- Similarly, we may not use more than 500h of machine 2 time:

$$\Longrightarrow \quad 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500$$

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- Producing x_i units of product $i \in \mathcal{P}$ requires

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units of skilled labour, and this must not exceed y_s.

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- ... and $y_s \leq 600$ as well as $y_u \leq 650$ as only limited amounts of labour can be purchased.

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- Revenue from sales:

$$300x_1 + 260x_2 + 220x_3 + 180x_4$$

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$$\begin{aligned} \text{maximize} \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 \\ & - 8y_s - 6y_u \end{aligned}$$

WaterTech – Entire Model

$$
\begin{aligned}
\max \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t.} \quad & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
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Solution (via CPLEX): $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$, $y_u = 650$ of profit \$$15433 + \frac{1}{3}$.

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- Similar: solution to word description is an assignment to the unknowns

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- It is easily checked that

$$x_1 = 10, x_2 = 50, x_3 = 0, x_4 = 20, y_s = y_u = 600$$

is feasible for the mathematical program we wrote.

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- In the example, the map was simply the identity. This need not necessarily be the case in general!

Module 1: Formulations (LP Models)

Constrained Optimization

In this course, we consider optimization problems of the following form:

$$\min\{f(x) \,:\, g_i(x) \leq b_i,\, (1 \leq i \leq m), x \in \mathbb{R}^n\},$$

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where

- $n, m \in \mathbb{N}$,
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This class: all functions are affine.

Modeling: Linear Programs

Affine Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = a^T x + \beta$ for $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$.

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(iii) $f(x) = 5x - 3\cos(x) + \sqrt{x}$

Affine Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = a^T x + \beta$ for $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$.
It is linear if in addition $\beta = 0$.

Example

(i) $f(x) = 2x_1 + 3x_2 - x_3 + 7$ (affine, but not linear)
(ii) $f(x) = -3x_1 + 5x_3$ (linear)
(iii) $f(x) = 5x - 3\cos(x) + \sqrt{x}$ (not affine and not linear)

Definition

The optimization problem

$$\min\{f(x) : g_i(x) \leq b_i, \quad \forall 1 \leq i \leq m, x \in \mathbb{R}^n\} \tag{P}$$

is called a linear program if f is affine and $g_1, \ldots, g_m$ is finite number of linear functions.

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Comments:

Instead of set notation, we often write LPs more verbosely

$$\begin{array}{rl} \max & -2x_1 + 7x_3 \\ \text{subject to} & x_1 + 7x_2 \le 3 \\ & 3x_2 + 4x_3 \le 2 \\ & x_1, x_3 \ge 0 \end{array}$$

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$$
\begin{aligned}
\max \quad & -1/x_1 - x_3 \\
\text{subject to} \quad & 2x_1 + x_3 < 3 \\
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\end{aligned}
$$

$(\alpha \in \mathbb{R})$

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$(\alpha \in \mathbb{R})$

Production revisited

$$
\begin{array}{ll}
\max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t.} & \begin{array}{rcr}
11x_1 + 7x_2 + 6x_3 + 5x_4 & \leq & 700 \\
4x_1 + 6x_2 + 5x_3 + 4x_4 & \leq & 500 \\
8x_1 + 5x_2 + 5x_3 + 6x_4 & \leq & y_s \\
7x_1 + 8x_2 + 7x_3 + 4x_4 & \leq & y_u \\
y_s & \leq & 600 \\
y_u & \leq & 650 \\
x_1, x_2, x_3, x_4, y_u, y_s & \geq & 0.
\end{array}
\end{array}
$$

The mathematical program for WaterTech example from last class is in fact an LP!

Example: Multiperiod Models

Main feature of WaterTech production model:
Decisions about production levels have to be made once and for all.

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In practice, we often have to make a sequence of decision that influence each other.

One such example: Multiperiod Models:

- Time is split into periods,
- We have to make a decision in each period, and
- All decisions influence the final outcome.

KW Oil

KW Oil is local supplier of heating oil

Needs to decide on how much oil to purchase in order to satisfy demand of its customers

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Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

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Price ($\$/\ell$)	0.75	0.72	0.92	0.90

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Question: When should we purchase how much oil?

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Additional complication: Company has storage tank that

- Currently (beginning of month 1) contains 2000 litres of oil, and
- Has a capacity of 4000 litres of oil.

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Additional complication: Company has storage tank that

- Currently (beginning of month 1) contains 2000 litres of oil, and
- Has a capacity of 4000 litres of oil.

Assumption: Oil is delivered at beginning of month, and consumption occurs mid month

KW Oil Model – Variables

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

Month	1	2	3	4
Price (\$/ℓ)	0.75	0.72	0.92	0.90

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(i) Need to decide how many litres of oil to purchase in each month i.
$\longrightarrow$ variable p_i for $i \in [4]$

(ii) How much oil is stored in the tank at beginning of month i?
$\longrightarrow$ variable t_i for $i \in [4]$

Objective function

Minimize cost of oil procurement.

Variables:

p_i : oil purchase in month i
t_i : tank level in month i

Objective function

Minimize cost of oil procurement.

$$\min \quad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

Variables:

p_i : oil purchase in month i
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Objective function

Minimize cost of oil procurement.

$$\min \quad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

Constraints: when do

$$t_1, \ldots, t_4, p_1, \ldots, p_4$$

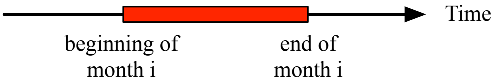
correspond to a feasible purchasing scheme?

Variables:

p_i : oil purchase in month i
t_i : tank level in month i

Constraints

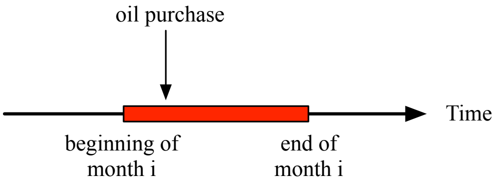
Assumptions:



Constraints

Assumptions:

(i) Oil is purchased at beginning of month



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Assumptions:

(i) Oil is purchased at beginning of month

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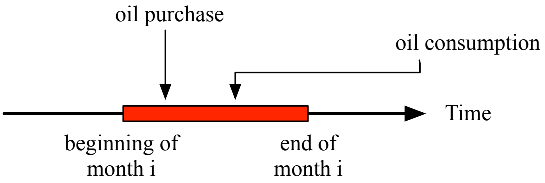

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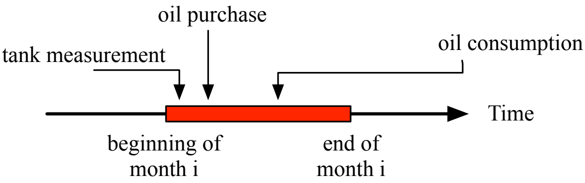

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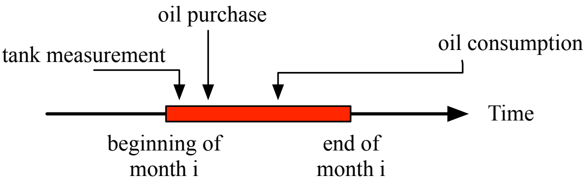

Variables:

p_i : oil purchase in month i
t_i : tank level in month i

We need: $p_i + t_i \geq \{\text{demand in month } i\}$

Constraints

Assumptions:

(i) Oil is purchased at beginning of month

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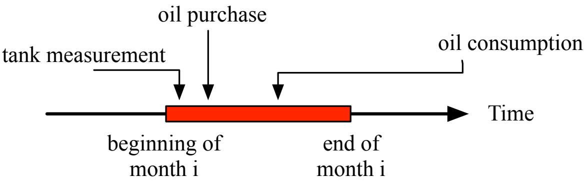

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[Balance Equation] $\quad p_i + t_i = \{\text{demand in month } i\} + t_{i+1}$

Constraints

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Tank content in month 1: 2000 litres

Constraints

[Balance Equation] $p_i + t_i =$ {demand in month i} $+ t_{i+1}$

Month 1:
$p_1 + 2000 = 5000 + t_2$

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

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Month 3:

$p_3 + t_3 = 9000 + t_4$

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Tank content in month 1: 2000 litres

Month 1:
$p_1 + 2000 = 5000 + t_2$

Month 2:
$p_2 + t_2 = 8000 + t_3$

Month 3:
$p_3 + t_3 = 9000 + t_4$

Month 4:
$p_4 + t_4 \geq 6000$

KW Oil: Entire LP

$$
\begin{array}{ll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 \\
\text{subject to} &
\end{array}
$$

$$
\begin{array}{rcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2,3,4) \\
t_i, p_i & \geq & 0 & (i = 1,2,3,4)
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t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array}
\end{array}
$$

Solution: $p = (3000, 12000, 5000, 6000)^T$, and
$t = (2000, 0, 4000, 0)^T$

KW Oil: Add-Ons

Can easily capture additional features. E.g. ...

> Storage comes at a cost: storage cost is \$.15 per litre/month.

$$
\begin{array}{lllll}
\min & 0.75p_1+0.72p_2+0.92p_3+0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1+t_1 & = & 5000+t_2 & \\
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Minimize maximum #l of oil purchased over all months

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(ii) Will have to add constraints

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& p_1 + t_1 & = & 5000 + t_2 & \\
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Goal: Minimize maximum #l of oil purchased over all months.

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

(ii) Add constraints

$$p_i \leq M$$

for all $i \in [4]$.

$$
\begin{array}{ll}
\text{min} & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 \\
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(iii) Done?

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for all $i \in [4]$.

(iii) Done? No! Need to replace objective function by

$$\min \quad M$$

$$\begin{array}{lllll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2, 3, 4) \\
& t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}$$

Goal: Minimize maximum #l of oil purchased over all months.

Minimizing the Maximum Purchase: LP

$$\min \quad M$$

s.t.

$$\begin{array}{lcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
p_i & \leq & M & (i = 1, 2, 3, 4) \\
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\end{array}$$

KW Oil: Correctness

Why is this a correct model?

$$
\begin{array}{lllll}
\min & M & & & \\
\text{s.t.} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
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\end{array}
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Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

$$\min \quad M$$

s.t.

$$\begin{array}{rcll}
p_1 + t_1 & = & 5000 + t_2 & \\
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Clearly: $M \geq \max_i p_i$

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KW Oil: Correctness

Why is this a correct model?

Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

Clearly: $M \geq \max_i p_i$

Since M, p, t is optimal we must have $M = \max_i p_i$. Why?

$$\min \quad M$$

s.t.

$$\begin{array}{rcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
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t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}$$

Module 1: Formulations (IP Models)

Recap: WaterTech

$$
\begin{aligned}
\max \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t.} \quad & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
& 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\
& 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\
& 7x_1 + 8x_2 + 7x_3 + 4x_4 \leq y_u \\
& y_s \leq 600 \\
& y_u \leq 650 \\
& x_1, x_2, x_3, x_4, y_u, y_s \geq 0.
\end{aligned}
$$

Recap: WaterTech

$$\begin{array}{rl} \max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\ \text{s.t.} & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\ & 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\ & 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\ & 7x_1 + 8x_2 + 7x_3 + 4x_4 \leq y_u \\ & y_s \leq 600 \\ & y_u \leq 650 \\ & x_1, x_2, x_3, x_4, y_u, y_s \geq 0. \end{array}$$

Optimal solution: $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$, $y_u = 650$

Recap: WaterTech

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& 7x_1 + 8x_2 + 7x_3 + 4x_4 \leq y_u \\
& y_s \leq 600 \\
& y_u \leq 650 \\
& x_1, x_2, x_3, x_4, y_u, y_s \geq 0.
\end{aligned}$$

Optimal solution: $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$, $y_u = 650$

Fractional solutions are often not desirable! Can we force solution to take on only integer values?

- Yes!

  An integer program is a linear program with added integrality constraints for some/all variables.

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 2x_4 \\
\text{s.t.} \quad & x_1 + x_2 \leq 1 \\
& -x_2 - x_3 \geq -1 \\
& x_1 + x_3 = 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

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& x_1, x_3 \text{ integer.}
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- We call an IP mixed if there are integer and fractional variables, and pure otherwise.

$$
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\max \quad & x_1 + x_2 + 2x_4 \\
\text{s.t.} \quad & x_1 + x_2 \le 1 \\
& -x_2 - x_3 \ge -1 \\
& x_1 + x_3 = 1 \\
& x_1, x_2, x_3 \ge 0 \\
& x_1, x_3 \text{ integer.}
\end{aligned}
$$

- Yes!

  An integer program is a linear program with added integrality constraints for some/all variables.

- We call an IP mixed if there are integer and fractional variables, and pure otherwise.

- Difference between LPs and IPs is subtle. Yet: LPs are easy to solve, IPs are not!

$$
\begin{aligned}
\max \quad & x_1 + x_2 + 2x_4 \\
\text{s.t.} \quad & x_1 + x_2 \leq 1 \\
& -x_2 - x_3 \geq -1 \\
& x_1 + x_3 = 1 \\
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- Integer programs are provably difficult to solve!

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- The running time of an algorithm is then the number of steps that an algorithm takes.

Can we solve IPs?

- Integer programs are provably difficult to solve!
- Every problem instance has a size which we normally denote by n.
  Think: $n \sim$ number of variables/constraints of IP.
- The running time of an algorithm is then the number of steps that an algorithm takes.
- It is stated as a function of n: $f(n)$ measures the largest number of steps an algorithm takes on an instance of size n.

Can we solve IPs?

- An algorithm is efficient if its running time $f(n)$ is a polynomial of n.

Can we solve IPs?

- An algorithm is efficient if its running time $f(n)$ is a polynomial of n.
- LPs can be solved efficiently.


The New York Times U.S.

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BREAKTHROUGH IN PROBLEM SOLVING

By JAMES GLEICK
Published: November 19, 1984

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The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

These problems are fiendishly complicated systems, often with

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Can we solve IPs?

- An algorithm is efficient if its running time $f(n)$ is a polynomial of n.
- LPs can be solved efficiently.
- IPs are very unlikely to admit efficient algorithms!

The New York Times   U.S.

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Can we solve IPs?

- An algorithm is efficient if its running time $f(n)$ is a polynomial of n.
- LPs can be solved efficiently.
- IPs are very unlikely to admit efficient algorithms!
- It is very important to look for an efficient algorithm for a problem. The table states actual running times of a computer that can execute 1 million operations per second on an instance of size $n = 100$:

The New York Times

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$f(n)$	n	$n\log_2(n)$	n^3	1.5^n	2^n

IP Models: Knapsack

KitchTech Shipping

- Company wishes to ship crates from Toronto to Kitchener.

KitchTech Shipping

- Company wishes to ship crates from Toronto to Kitchener.
- Each crate type has weight and value:

Type	1	2	3	4	5	6
weight (lbs)	30	20	30	90	30	70
value ($)	60	70	40	70	20	90

KitchTech Shipping

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- Total weight of crates shipped must not exceed 10,000 lbs.

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Type	1	2	3	4	5	6
weight (lbs)	30	20	30	90	30	70
value ($)	60	70	40	70	20	90

- Total weight of crates shipped must not exceed 10,000 lbs.
- Goal: Maximize total value of shipped goods.

IP Model

- Variables.

IP Model

- Variables. One variable x_i for number of crates of type i to pack.

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- Objective function: Maximize total value.

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$$30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000$$

- Objective function: Maximize total value.

$$\max \quad 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6$$

IP Model

$$\begin{aligned} \max \quad & 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6 \\ \text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000 \\ & x_i \geq 0 \quad (i \in [6]) \\ & x_i \text{ integer} \quad (i \in [6]) \end{aligned}$$

IP Model

$$
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\max \quad & 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6 \\
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& x_i \geq 0 \quad (i \in [6]) \\
& x_i \text{ integer} \quad (i \in [6])
\end{aligned}
$$

Let's make this model a bit more interesting...

KitchTech: Added Conditions

Suppose that ...

- We must not send more than 10 crates of the same type.

$$
\begin{aligned}
\max \quad & 60x_1 + 70x_2 + 40x_3 + \\
& 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\
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& 0 \leq x_i \leq 10 \quad (i \in [6]) \\
& x_i \text{ integer} \quad (i \in [6])
\end{aligned}
$$

KitchTech: Added Conditions

Suppose that ...

- We must not send more than 10 crates of the same type.
- Can only send crates of type 3, if we send at least 1 crate of type 4.

$$\begin{aligned} \max \quad & 60x_1 + 70x_2 + 40x_3 + \\ & 70x_4 + 20x_5 + 90x_6 \\ \text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\ & 90x_4 + 30x_5 + 70x_6 \leq 10000 \\ & 0 \leq x_i \leq 10 \quad (i \in [6]) \\ & x_i \text{ integer} \quad (i \in [6]) \end{aligned}$$

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  Note: Can send at most 10 crates of type 3 by previous constraint!

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KitchTech: Added Conditions

Correctness:

- $x_4 \geq 1 \longrightarrow$ new constraint is redundant!

$$
\begin{aligned}
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& 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\
& 90x_4 + 30x_5 + 70x_6 \leq 10000 \\
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& x_i \text{ integer} \quad (i \in [6])
\end{aligned}
$$

KitchTech: Added Conditions

Correctness:

- $x_4 \geq 1 \longrightarrow$ new constraint is redundant!
- $x_4 = 0 \longrightarrow$ new constraint becomes

$$x_3 \leq 0.$$

$$\begin{aligned}
\max \quad & 60x_1 + 70x_2 + 40x_3 + \\
& 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\
& 90x_4 + 30x_5 + 70x_6 \leq 10000 \\
& x_3 \leq 10x_4 \\
& 0 \leq x_i \leq 10 \quad (i \in [6]) \\
& x_i \text{ integer} \quad (i \in [6])
\end{aligned}$$

KitchTech: One More Tricky Case

Suppose that we must

1. take a total of at least 4 crates of type 1 or 2, or
2. take at least 4 crates of type 5 or 6.

$$
\begin{aligned}
\max \quad & 60x_1 + 70x_2 + 40x_3 + \\
& 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\
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Ideas?

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KitchTech: One More Tricky Case

Suppose that we must

1. take a total of at least 4 crates of type 1 or 2, or
2. take at least 4 crates of type 5 or 6.

Ideas?

Create a new variable y s.t.

1. $y = 1 \longrightarrow x_1 + x_2 \geq 4$,
2. $y = 0 \longrightarrow$

$$
\begin{aligned}
\max \quad & 60x_1 + 70x_2 + 40x_3 + \\
& 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\
& 90x_4 + 30x_5 + 70x_6 \leq 10000 \\
& x_3 \leq 10x_4 \\
& 0 \leq x_i \leq 10 \quad (i \in [6]) \\
& x_i \text{ integer} \quad (i \in [6])
\end{aligned}
$$

KitchTech: One More Tricky Case

Create a new variable y s.t.

1. $y = 1 \longrightarrow x_1 + x_2 \geq 4$,
2. $y = 0 \longrightarrow x_5 + x_6 \geq 4$.

Force y to take on values 0 or 1.

Add constraints:

$$
\begin{array}{rl}
\max & 60x_1 + 70x_2 + 40x_3 + \\
 & 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} & 30x_1 + 20x_2 + 30x_3 + \\
 & 90x_4 + 30x_5 + 70x_6 \leq 10000 \\
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 & 0 \leq x_i \leq 10 \quad (i \in [6]) \\
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\end{array}
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Create a new variable y s.t.

1. $y = 1 \longrightarrow x_1 + x_2 \geq 4$,
2. $y = 0 \longrightarrow x_5 + x_6 \geq 4$.

Force y to take on values 0 or 1.

Add constraints:

1. $x_1 + x_2 \geq 4y$

$$
\begin{aligned}
\max \quad & 60x_1 + 70x_2 + 40x_3 + \\
& 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\
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\end{aligned}
$$

KitchTech: One More Tricky Case

Create a new variable y s.t.

1. $y = 1 \longrightarrow x_1 + x_2 \geq 4$,
2. $y = 0 \longrightarrow x_5 + x_6 \geq 4$.

Force y to take on values 0 or 1.

Add constraints:

1. $x_1 + x_2 \geq 4y$
2. $x_5 + x_6 \geq 4(1 - y)$

$$
\begin{array}{rl}
\max & 60x_1 + 70x_2 + 40x_3 + \\
 & 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} & 30x_1 + 20x_2 + 30x_3 + \\
 & 90x_4 + 30x_5 + 70x_6 \leq 10000 \\
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Create a new variable y s.t.

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2. $y = 0 \longrightarrow x_5 + x_6 \geq 4$.

Force y to take on values 0 or 1.

Add constraints:

1. $x_1 + x_2 \geq 4y$
2. $x_5 + x_6 \geq 4(1 - y)$
3. $0 \leq y \leq 1$

$$
\begin{aligned}
\max \quad & 60x_1 + 70x_2 + 40x_3 + \\
& 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} \quad & 30x_1 + 20x_2 + 30x_3 + \\
& 90x_4 + 30x_5 + 70x_6 \leq 10000 \\
& x_3 \leq 10x_4 \\
& 0 \leq x_i \leq 10 \quad (i \in [6]) \\
& x_i \text{ integer} \quad (i \in [6])
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$$

KitchTech: One More Tricky Case

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& y \text{ integer} \\
& x_i \text{ integer} \quad (i \in [6])
\end{aligned}
$$

Binary Variables

Variable y is called a binary variable.

These are very useful for modeling logical constraints of the form:

[Condition (A or B) and C] $\longrightarrow$ D

Will see more examples ...

$$\begin{aligned}
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& 0 \leq y \leq 1 \\
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IP Models: Scheduling

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3	5	9	2	7

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Can we solve this using IP?

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3. Constraints. Need to ensure that enough people work on each of the days.

   Question: Given a solution $(x_M, x_T, x_W, x_{Th}, x_F)$, how many people work on Monday?
   All but those that start on Tuesday; i.e.,

$$x_M + x_W + x_{Th} + x_F.$$

Constraints

[Daily Demand]

Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7

Monday:

$$x_M + x_W + x_{Th} + x_F \geq 3$$

Constraints

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Mon	Tues	Wed	Thurs	Fri
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Monday: $x_M + x_W + x_{Th} + x_F \geq 3$

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Constraints

[Daily Demand]

Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7

Monday: $x_M + x_W + x_{Th} + x_F \geq 3$

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Thursday: $x_M + x_T + x_W + x_T \geq 2$

Constraints

[Daily Demand]

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Tuesday: $x_M + x_T + x_{Th} + x_F \geq 5$

Wednesday: $x_M + x_T + x_W + x_F \geq 9$

Thursday: $x_M + x_T + x_W + x_T \geq 2$

Friday: $x_T + x_W + x_{Th} + x_F \geq 7$

Scheduling LP

$$
\begin{array}{rl}
\min & x_M + x_T + x_W + x_{Th} + x_F \\
\text{s.t.} & x_M + x_W + x_{Th} + x_F \geq 3 \\
& x_M + x_T + x_{Th} + x_F \geq 5 \\
& x_M + x_T + x_W + x_F \geq 9 \\
& x_M + x_T + x_W + x_T \geq 2 \\
& x_T + x_W + x_{Th} + x_F \geq 7 \\
& x \geq \nvdash, x \text{ integer}
\end{array}
$$

Quiz

Given an integer program with integer variables $x_1, \ldots, x_6$. Let

$$\mathcal{S} := \{127, 289, 1310, 2754\}.$$

We want to add constraints and/or variables to the IP that enforce that the $x_1 + \ldots + x_6$ is in $\mathcal{S}$. How?

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- Want: Exactly one of these variables is 1 in a feasible solution.
- If $y_n = 1$ for $n \in \mathcal{S}$ then $\sum_{i=1}^{6} x_i = n$

Quiz

Add the following constraints:

$$y_{127} + y_{289} + y_{1310} + y_{2754} = 1$$

$$\sum_{i=1}^{6} x_i = \sum_{i \in \mathcal{S}} i y_i$$

$$0 \leq y_i \leq 1, \quad y_i \text{ integer} \quad \forall i \in \mathcal{S}$$

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Why is the resulting IP correct?

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- An integer program is obtained by adding integrality constraints for some/all of the variables to an LP.

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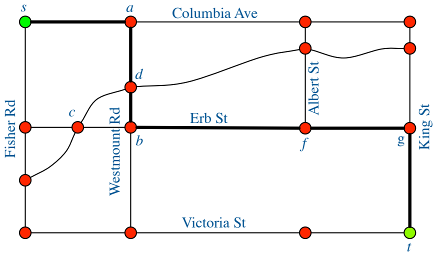
- Variables that can take value 0 or 1 only are called binary.
- Binary variables are useful for expressing logical conditions.

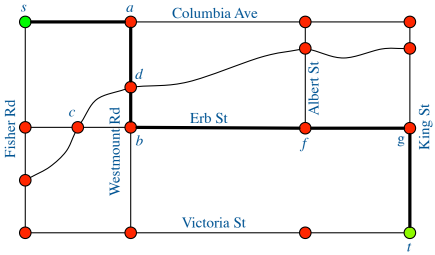
Module 1: Formulations (Optimization on Graphs)

Shortest Paths



- Familiar problem: Starting at location s, we wish to travel to t.
  What is the best (i.e., shortest) route?

Shortest Paths



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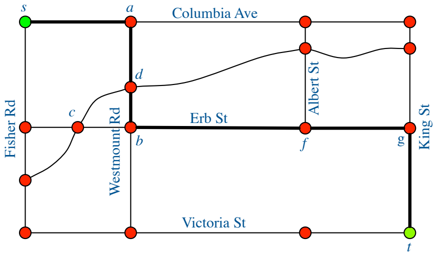
- In the figure above, such a route is indicated in bold.

Shortest Paths



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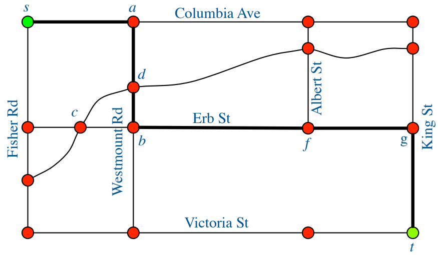

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Graph Theory 101

Rephrasing this problem in the language of graph theory helps!

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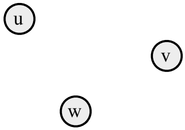

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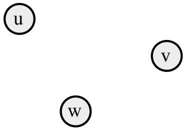

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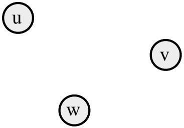

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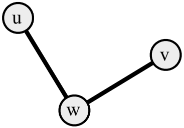

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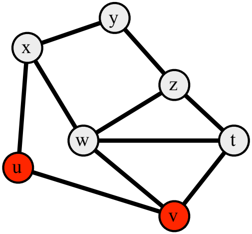
Two vertices u and v are adjacent if $uv \in E$.



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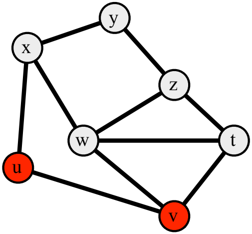

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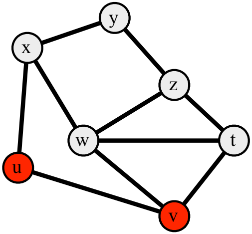

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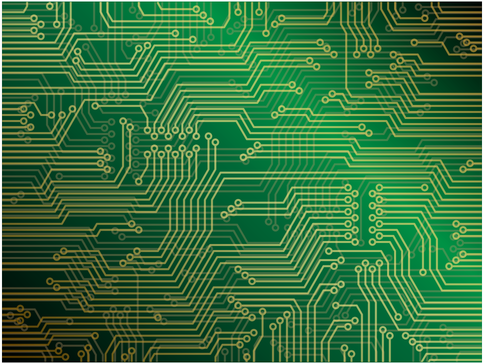

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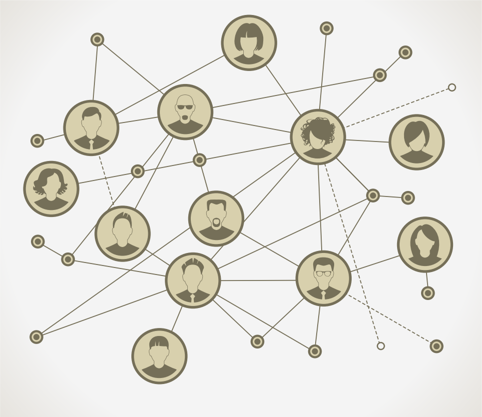

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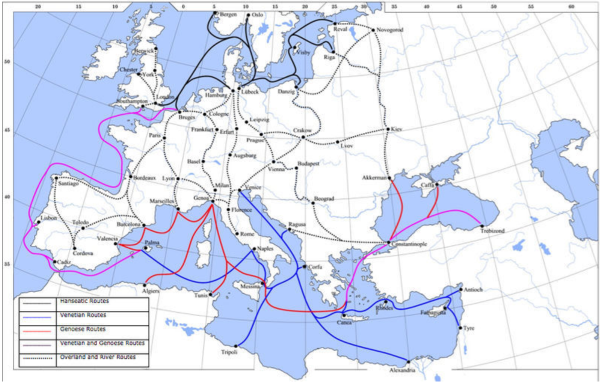

Lampman, 2008 [Online Image]. Late Medieval Trade Routes. Wikimedia Commons.
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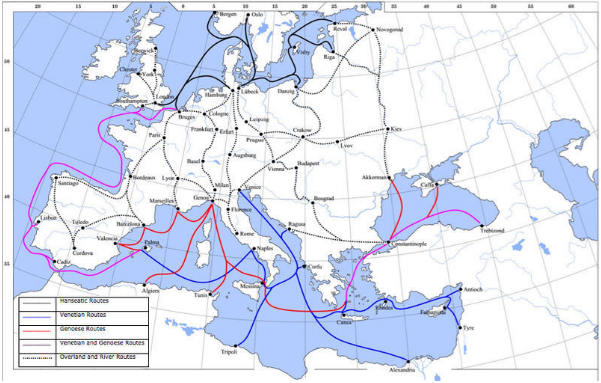
- .... and many more!



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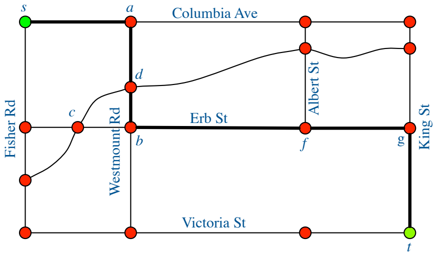
The Map of Water Town



- We can think of the street map as a graph, G.

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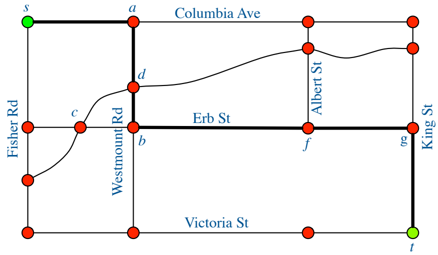

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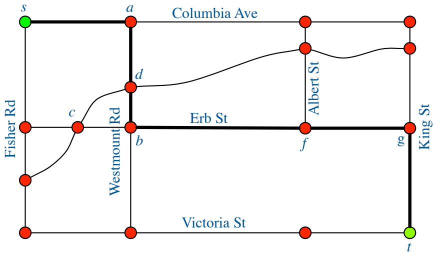

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- Edges: Road segments connecting adjacent intersections

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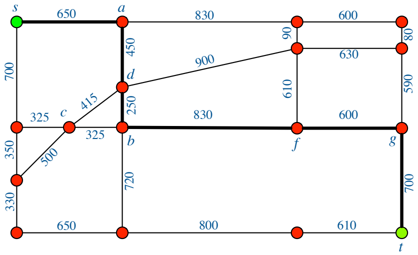

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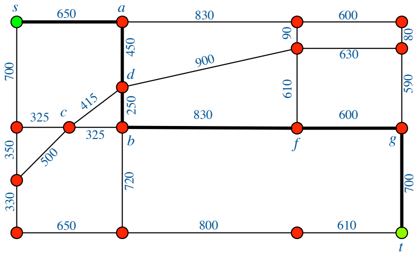
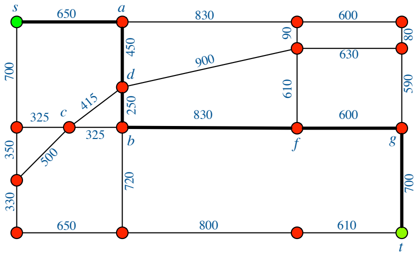


- Each edge $e \in E$ is labelled by its length $c_e \geq 0$.
- We are looking for a path connecting s and t of smallest total length!

Paths



An s, t-path in $G = (V, E)$ is a sequence

$$v_1v_2, v_2v_3, v_3v_4, \ldots, v_{k-2}v_{k-1}, v_{k-1}v_k$$

where

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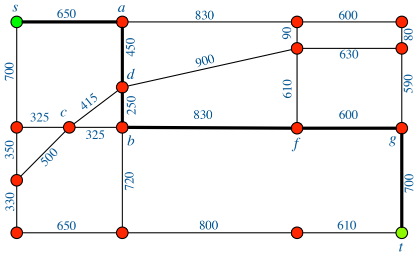

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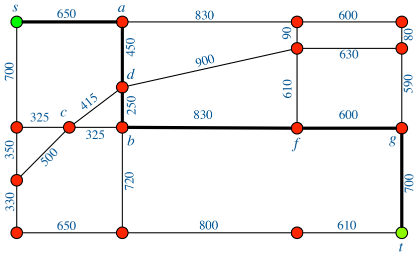

An s, t-path in $G = (V, E)$ is a sequence

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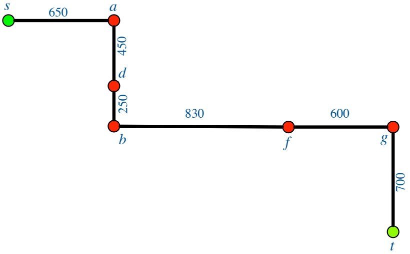

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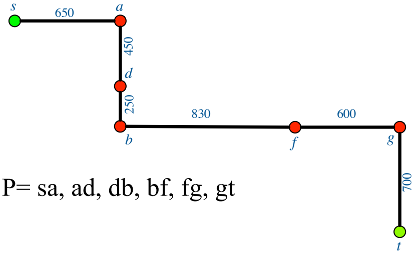

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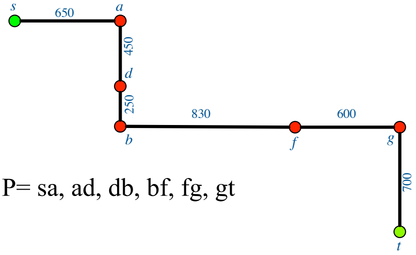

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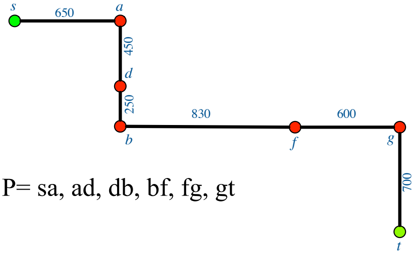

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$$\begin{aligned} c(P) &= c_{sa} + c_{ad} + c_{db} + c_{bf} + c_{fg} + c_{gt} \\ &= 650 + 490 + 250 + 830 + 600 + 700 \\ &= 3520 \end{aligned}$$

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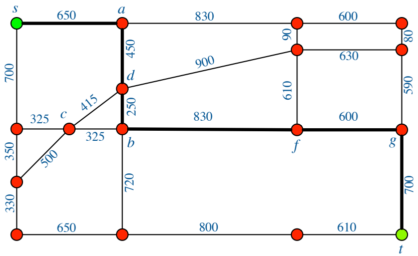

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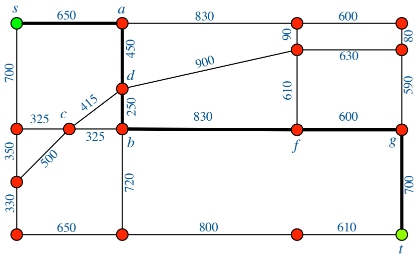

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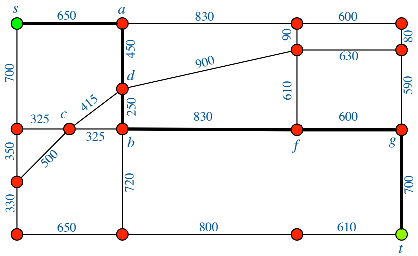

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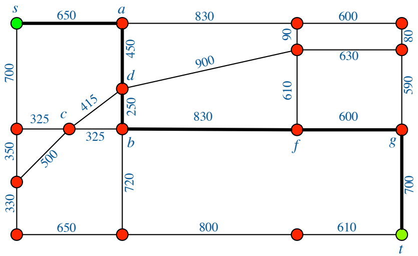

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$\longrightarrow$ Later!

Example: Matchings

WaterTech - Job Assignment

WaterTech has a collection of
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$\longrightarrow$ We will rephrase this in the language of graphs

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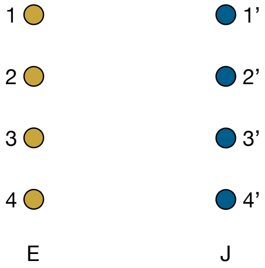

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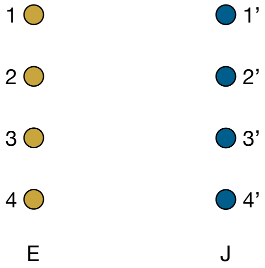

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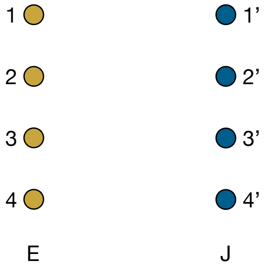

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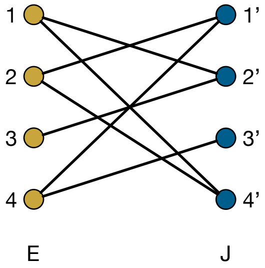

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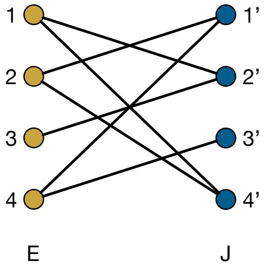

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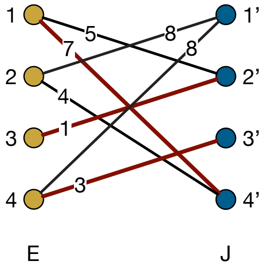

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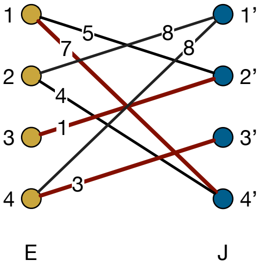

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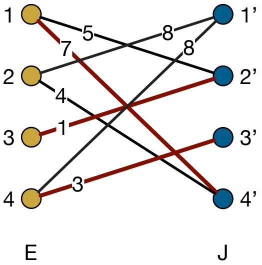

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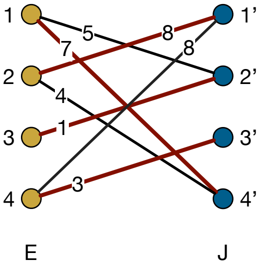

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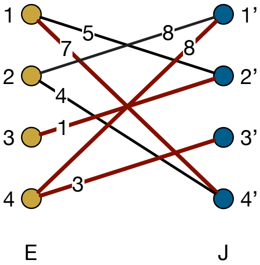

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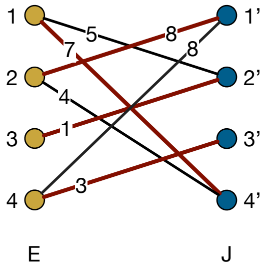

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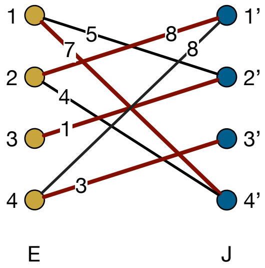

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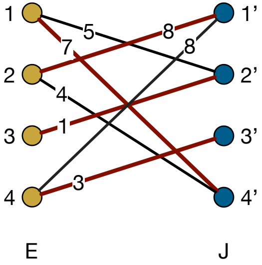

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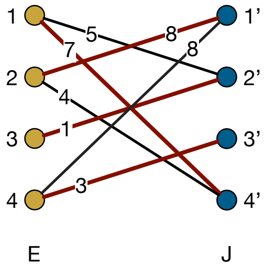

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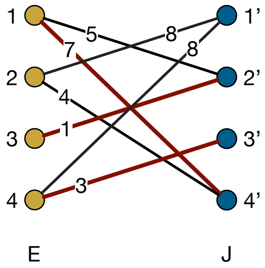

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Restating the Assignment Problem

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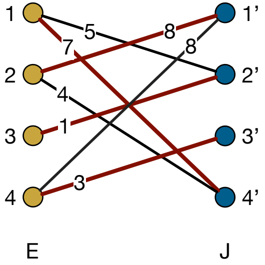
Note: Perfect matchings correspond to feasible assignments of workers to jobs!



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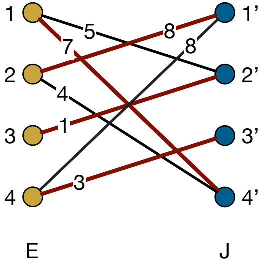

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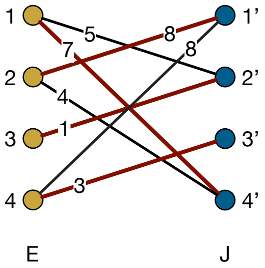

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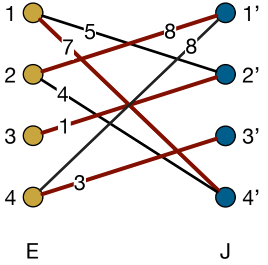

Restatement of original question:

Find a perfect matching M in our graph of smallest cost.

A Little More Notation...

Notation: Use $\delta(v)$ to denote the set of edges incident to v; i.e.,

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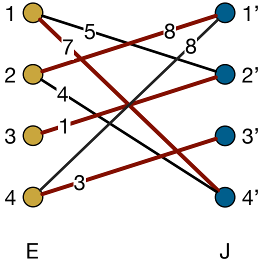

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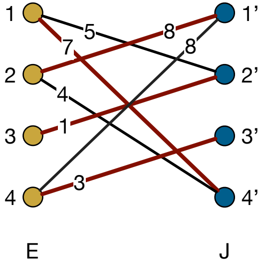

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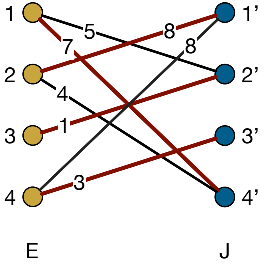

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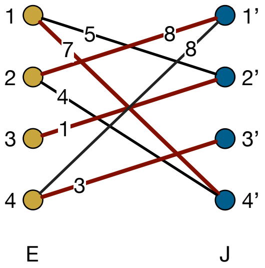

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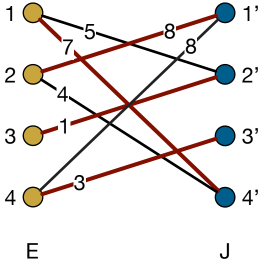

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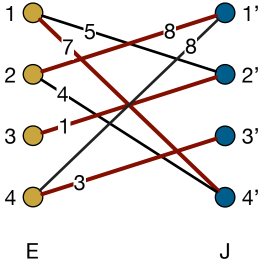

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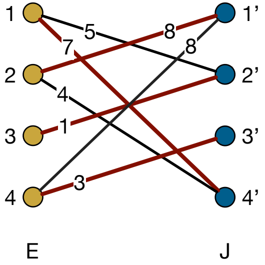

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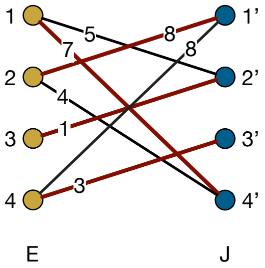
Constraints: For all $v \in V$, need

$$\sum (x_e \,:\, e \in \delta(v)) = 1$$



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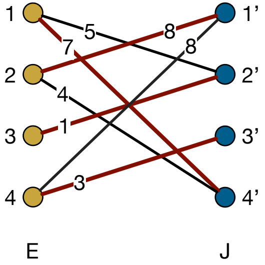
The IP will have a binary variable x_e for every edge $e \in E$. Idea:

$$x_e = 1 \quad \leftrightarrow \quad e \in M$$

Constraints: For all $v \in V$, need

$$\sum (x_e \,:\, e \in \delta(v)) = 1$$

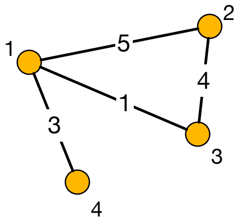


Objective:

$$\sum (c_e x_e \,:\, e \in E)$$

An IP for Perfect Matchings

$$\begin{aligned} \min\ & \sum (c_e x_e \,:\, e \in E) \\ \text{s.t.}\ & \sum (x_e \,:\, e \in \delta(v)) = 1 \ (v \in V) \\ & x \geq \mathbb{0},\ x \text{ integer} \end{aligned}$$



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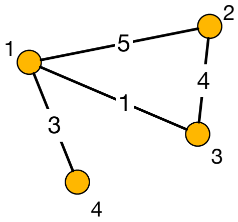

$$\min \quad (5,1,3,4)x$$

$$\text{s.t.} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 12 & 13 & 14 & 23 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} x = \mathbb{1}$$

$$x \geq \mathbb{0} \quad \text{integer}$$

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Module 1: Formulations (Shortest Paths)

Recap: Shortest Paths

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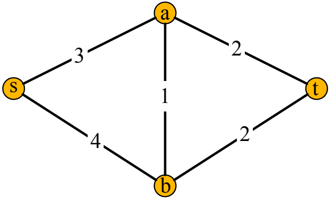

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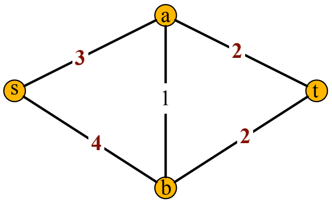

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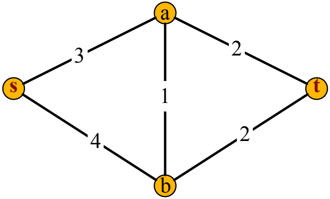

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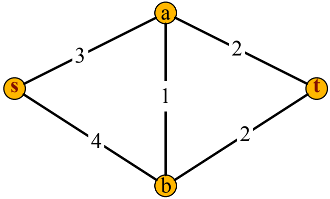

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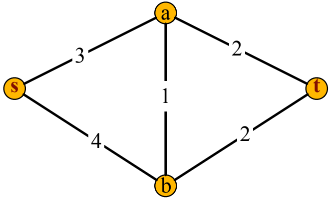

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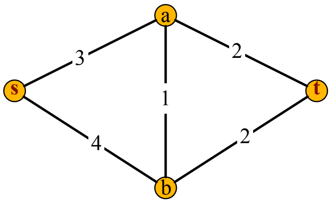

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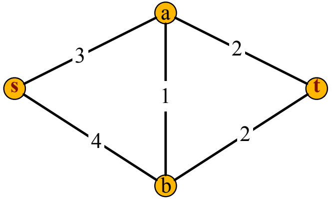

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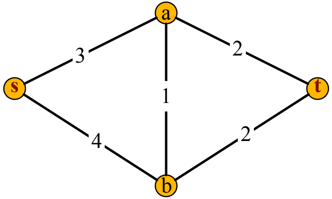

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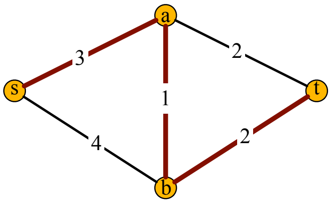

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E.g., $P = sa,\ ab,\ bt$

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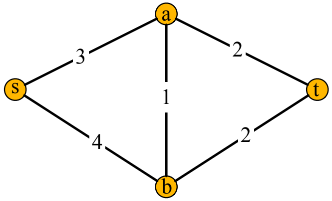

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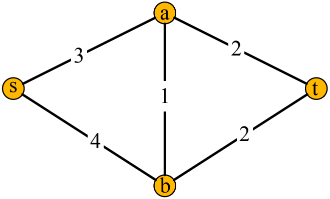

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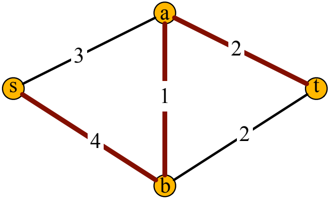

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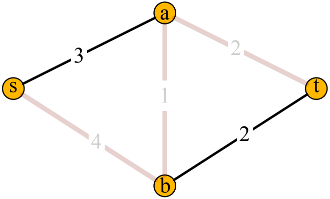

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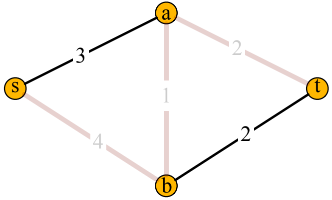

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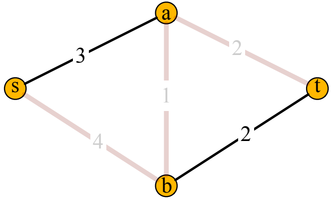

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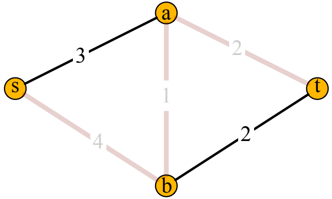

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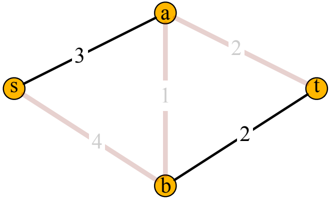

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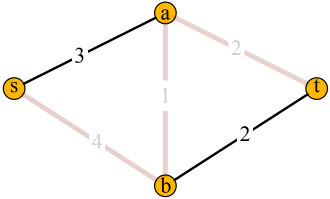

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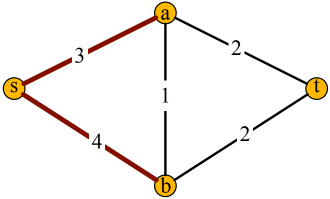

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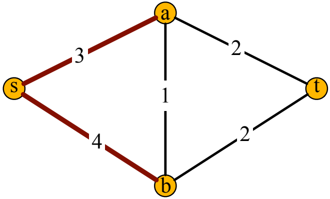

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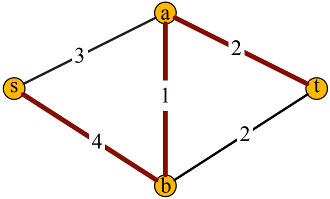

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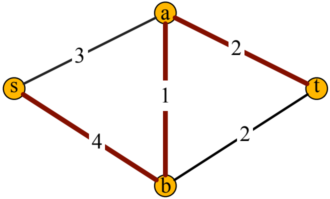

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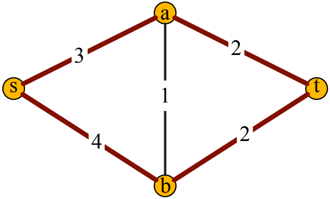

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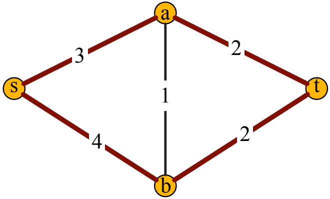

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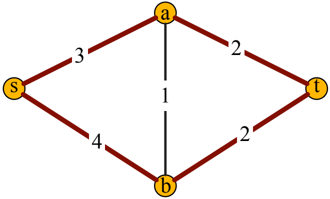

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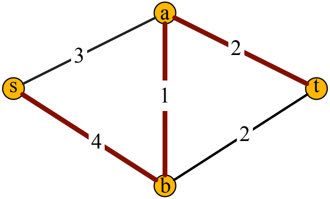

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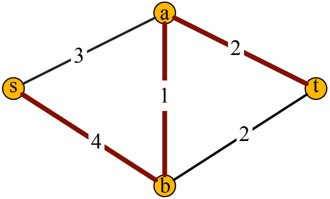

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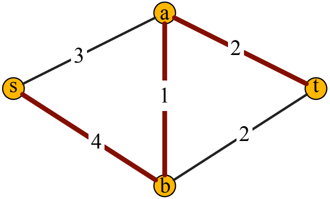

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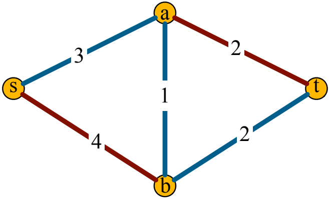

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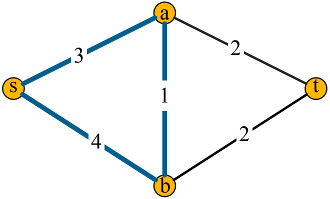

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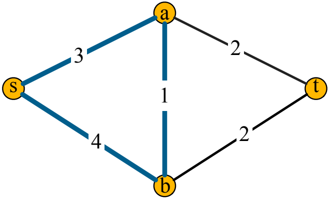

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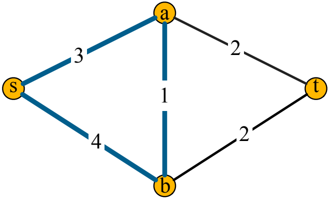

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- Suppose S has an edge from every s, t-cut, but S has no s, t-path.
- Let R be the set of vertices reachable from s in S:

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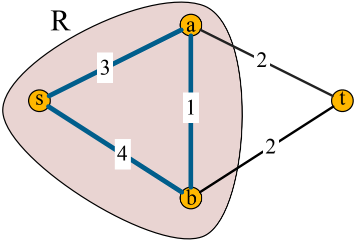

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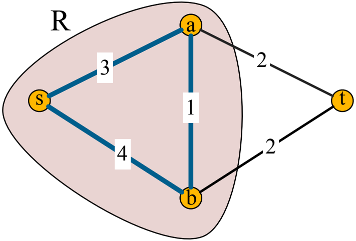

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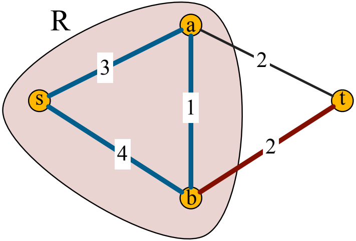

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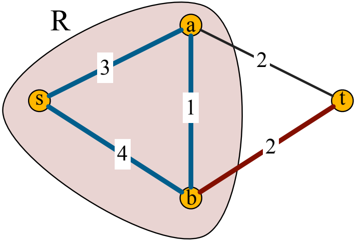

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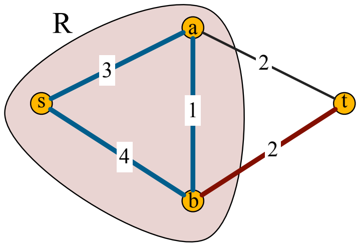

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$\longrightarrow \delta(R) \cap S = \emptyset$.
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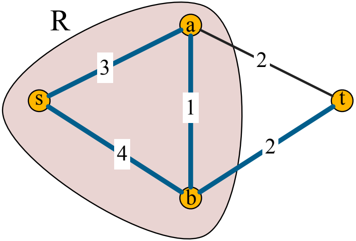
An IP for Shortest Paths

Variables: We have one binary variable x_e for each edge $e \in E$.



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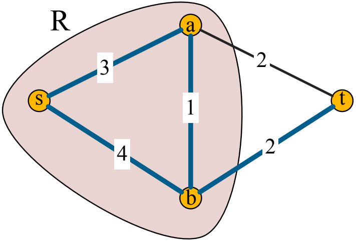

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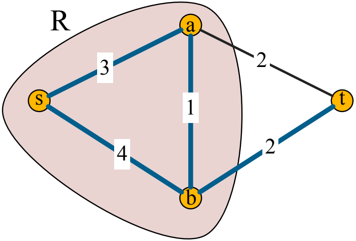

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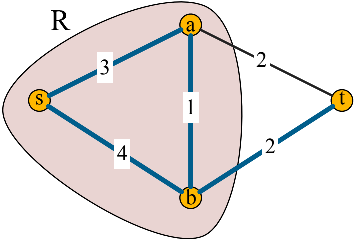

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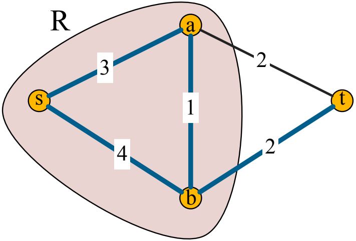

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$$\min \quad (3,4,1,2,2)x$$

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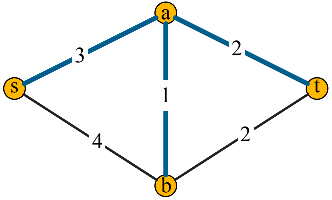

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For a binary solution x, define

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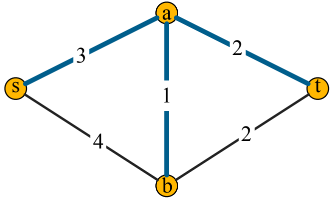

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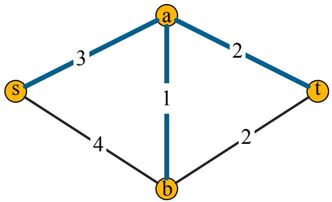

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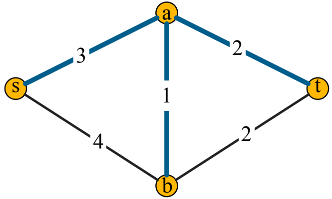

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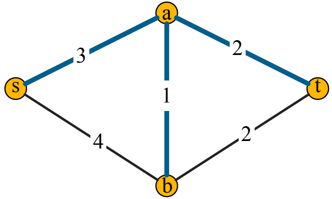

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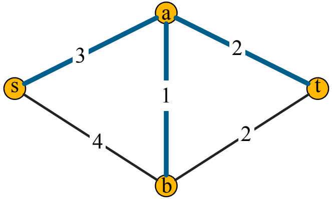

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If x is an optimal solution for the above IP and $c_e > 0$ for all $e \in E$, then S_x contains the edges of a shortest s,t-path.

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- Feasible solutions to the shortest path LP correspond to edge-sets that intersect every s, t-cut; optimal solutions are minimal in this respect if $c_e > 0$ for all $e \in E$.

Module 1: Formulations (Nonlinear Models)

So far ...

- Linear programs, and

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq \mathbb{0} \end{aligned}$$

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Both have linear/affine constraints.

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Now: Nonlinear generalization!

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Note: Linear programs are NLPs!

Example 1: Finding Close Points in an LP

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Problem: we are given an LP (P), and an infeasible point $\bar{x}$.

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Example 2: Binary IP via NLP

NLPs and Binary IPs

Suppose we are given a binary IP (i.e., an integer program all of whose variables are binary).

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Recall: (binary) IPs are generally hard to solve!

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Ideas?

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq \mathbb{0} \\ & x_j(1 - x_j) = 0 \quad (j \in [n]) \qquad (\star) \end{aligned}$$

NLPs and Binary IPs

Suppose we are given a binary IP (i.e., an integer program all of whose variables are binary).

Recall: (binary) IPs are generally hard to solve!

Now: can write any binary IP as an NLP!

Ideas?

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Q: Can you change the NLP to express the fact that x_j is any non-negative integer instead of binary?

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$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge \mathbb{0} \\ & \sin(\pi\, x_j) = 0 \quad (j \in [n]) \qquad (*) \end{aligned}$$

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Correctness: note that $\sin(\pi\, x_j) = 0$ only if x_j is an integer.

Example 3: Fermat's Last Theorem

Fermat's Last Theorem

Conjecture [Fermat, 1637]

There are no integers $x, y, z \geq 1$ and $n \geq 3$ such that

$$x^n + y^n = z^n.$$

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Arithmeticorum Liber II.

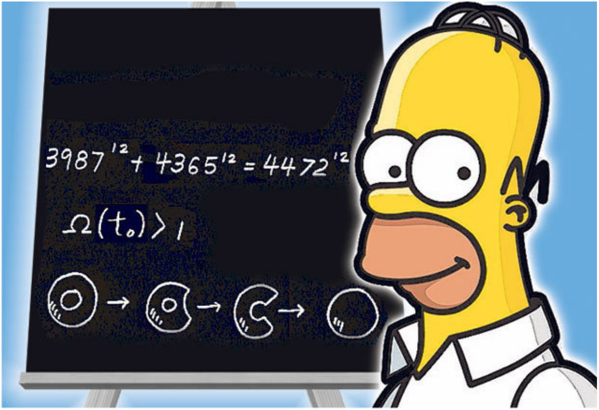
OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

This is false ...

This is false ...



... doh!

Fermat's Last Theorem

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There are no integers x, y, $z \geq 1$ and $n \geq 3$ such that

$$x^n + y^n = z^n.$$

Fermat’s Last Theorem

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In the margins of a copy of a 1670 article of Diophantus Arithmetica he wrote that he had a proof that was a bit too large to fit.

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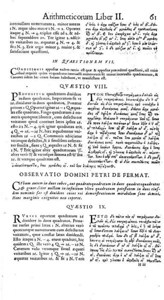

Arithmeticorum Liber II. 61

IN QVÆSTIONEM VII.

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

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Some 358 years later, Sir Andrew Wiles gave the first accepted proof of the theorem.

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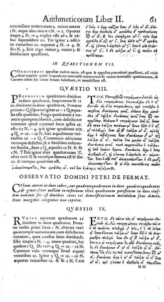

Arithmeticorum Liber II. 61

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QVÆSTIO IX.

NLP for Fermat's Last Theorem

$$\begin{aligned} \min \quad & (x_1^{x_4} + x_2^{x_4} - x_3^{x_4})^2 \\ & + (\sin \pi\, x_1)^2 + (\sin \pi\, x_2)^2 + (\sin \pi\, x_3)^2 + (\sin \pi\, x_4)^2 \end{aligned}$$

$$\begin{aligned} \text{s.t.} \quad & x_i \geq 1 \quad (i = 1 \ldots 3) \\ & x_4 \geq 3 \end{aligned}$$

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- The NLP is trivially feasible, and

NLP for Fermat's Last Theorem

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- The NLP is trivially feasible, and
- the value of any feasible solution is non-negative as its objective is the sum of squares.

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- In fact, the value of a solution (x_1, x_2, x_3, x_4) is 0 iff

NLP for Fermat's Last Theorem

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\end{aligned}
$$

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NLP for Fermat's Last Theorem

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Remark

Fermat's Last Theorem is true iff the NLP has optimal value greater than 0.

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Note: well known that there is an infinite sequence of feasible solutions whose objective value converges to 0!

NLP for Fermat's Last Theorem

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Remark

Fermat's Last Theorem is true iff the NLP has optimal value greater than 0.

Note: well known that there is an infinite sequence of feasible solutions whose objective value converges to 0!

Proving Fermat's Last Theorem amounts to showing that the value 0 can not be attained!

Recap

- Non-linear programs are of the form

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & \dots \\ & g_m(x) \leq 0, \end{aligned}$$

  where $f, g_1, \dots, g_m$ are non-linear functions.

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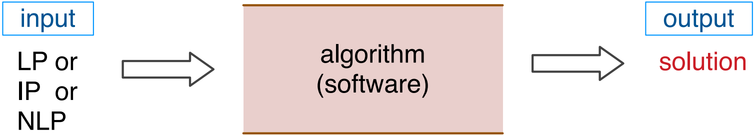
- Non-linear programs are strictly more general than integer programs, and thus likely difficult to solve.

- Some famous questions in Math can easily be reduced to solving certain NLPs

Module 2: Linear programs (Possible outcomes)

What does solving an optimization problem mean?



What does solving an optimization problem mean?

input		algorithm (software)		output
LP or IP or NLP	$\Rightarrow$		$\Rightarrow$	solution

$$
\begin{array}{ll}
\max & 2x_1 - 3x_2 \\
\text{s.t.} & \\
& x_1 + x_2 \le 1 \\
& x_1, x_2 \ge 0
\end{array}
$$

What does solving an optimization problem mean?

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LP or
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$x_1 = 1,$
$x_2 = 0$

Optimal
Solution

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$$x_1 = 1, \quad x_2 = 0$$

Optimal Solution

Remark

Sometimes the answer is not so straightforward!!!

Definition

An assignment of values to each of the variables is a feasible solution if all the constraints are satisfied.

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$$\max \quad x_1 - x_2$$

s.t.

$$\begin{array}{rcrcr} 4x_1 & + & x_2 & \geq & 2 \\ 2x_1 & - & 3x_2 & \leq & 4 \end{array}$$

$$x_1,\ x_2 \geq 0$$

Definition

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\begin{array}{ll}
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\text{s.t.} & \\
& \begin{array}{rcrcr} 4x_1 & + & x_2 & \geq & 2 \\ 2x_1 & - & 3x_2 & \leq & 4 \end{array} \\
& x_1,\ x_2 \geq 0
\end{array}
}
$$

$$x_1 = 1$$

$$x_2 = 3$$

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\end{array}
$$

$x_1 = 1$

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Feasible solution

$\Downarrow$

Problem is feasible

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$$x_1,\ x_2 \geq 0$$

$x_1 = 3$

$x_2 = 0$

NOT
feasible solution

But problem is
feasible.

Definition

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$$\boxed{\begin{array}{ll} \max & x_1 \\ \text{s.t.} & \\ & x_1 \leq 1 \\ & x_2 \geq 1 \end{array}}$$

$x_1 = 1, x_2 = \alpha$ optimal for all $\alpha \geq 1$.

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$x_1 = 1, x_2 = \alpha$ optimal for all $\alpha \geq 1$.

Remark

An optimization problem can have several optimal solutions.

Question

Does the following linear program have an optimal solution?

$$
\begin{array}{lrcl}
\max & x_1 & & \\
\text{s.t.} & & & \\
& x_1 & \geq & 2 \\
& x_1 & \leq & 1
\end{array}
$$

Question

Does the following linear program have an optimal solution?

$$
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\end{array}
}
$$

Infeasible problem, so no optimal solution

Question

Does the following linear program have an optimal solution?

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Question

Does every feasible optimization problem have an optimal solution?

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Infeasible problem, so no optimal solution

Question

Does every feasible optimization problem have an optimal solution? NO

$$\begin{array}{lr} \max & x_1 \\ \text{s.t.} & \\ & x_1 \geq 1 \end{array}$$

Question

Does the following linear program have an optimal solution?

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Infeasible problem, so
no optimal solution

Question

Does every feasible optimization problem have an optimal solution? NO

$$\boxed{\begin{array}{ll} \max & x_1 \\ \text{s.t.} & \\ & x_1 \geq 1 \end{array}}$$

Feasible ($x_1 = 1$),
but still no optimal solution!!!

Definition

- A maximization problem is unbounded if for every value M there exists a feasible solution with objective value greater than M.

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$\Longrightarrow$

We have seen three possible outcomes for an optimization problem:

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We have seen three possible outcomes for an optimization problem:

- It has an optimal solution

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- It has an optimal solution
- It is infeasible

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Question

Can anything else happen?

Definition

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We have seen three possible outcomes for an optimization problem:

- It has an optimal solution
- It is infeasible
- It is unbounded

Question

Can anything else happen? YES

Consider,

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

Consider,

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

- Feasible: set $x = 0$.

Consider,

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

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Proof

Suppose for a contradiction x is optimal solution.

Consider,

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

- Feasible: set $x = 0$.
- Not unbounded: 1 is an upper bound.
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Proof

Suppose for a contradiction x is optimal solution. Let

$$x' := \frac{x+1}{2}.$$

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$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

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Proof

Suppose for a contradiction x is optimal solution. Let

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Then $x' < 1$ feasible.

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Proof

Suppose for a contradiction x is optimal solution. Let

$$x' := \frac{x+1}{2}.$$

Then $x' < 1$ feasible. Moreover, $x' > x$.

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Thus x not optimal, contradiction.

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Any other example without strict inequalities?

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Any other example without strict inequalities? YES

Consider,

$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

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- Feasible: set $x = 1$.

Consider,

$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

- Feasible: set $x = 1$.
- Not unbounded: 0 is a lower bound.

Consider,

$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

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- Feasible: set $x = 1$.
- Not unbounded: 0 is a lower bound.
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Exercise

Check this optimization problem has no optimal solution.

$$\boxed{\begin{aligned} \max \quad & x \\ \text{s.t.} \quad & \\ & x < 1 \end{aligned}}$$

Not a linear program

Strict inequality

$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

Not a linear program

Objective function non-linear

$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

Not a linear program

Objective function non-linear

Remark

Linear programs are nicer than general optimization problems.

$$\boxed{\begin{aligned} \min \quad & \frac{1}{x} \\ \text{s.t.} \quad & \\ & x \geq 1 \end{aligned}}$$

Not a linear program
Objective function non-linear

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Linear programs are nicer than general optimization problems.

Fundamental theorem of linear programming

For any linear program one of the following holds:

- It has an optimal solution
- It is infeasible
- It is unbounded

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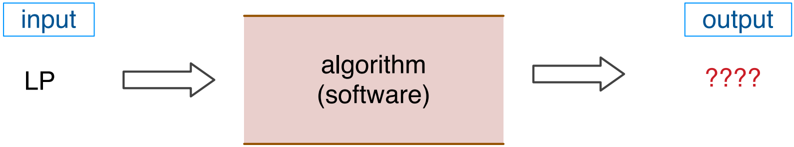
We will prove it later in the course.

We can now describe what we mean by solving a linear program,



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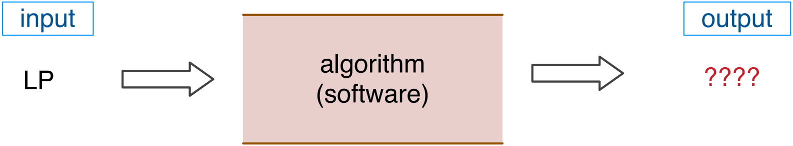

LP has an optimal solution

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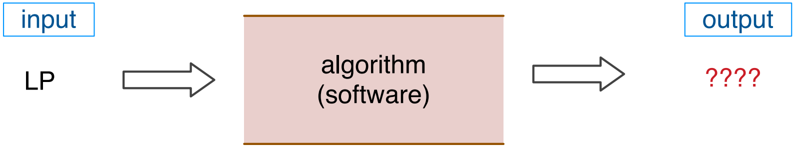

$\boxed{\text{LP has an optimal solution}}$

Return an optimal solution $\bar{x}$

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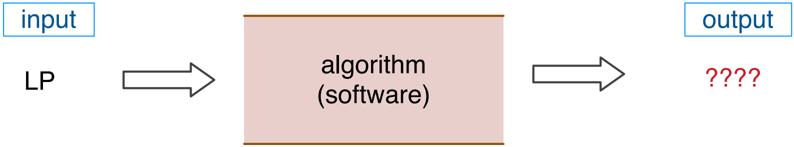

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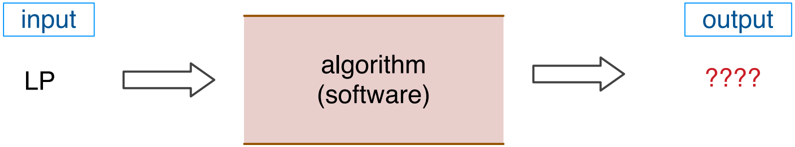

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LP is infeasible.

Say the LP is infeasible

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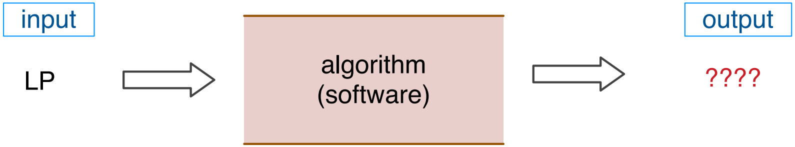

$\boxed{\text{LP has an optimal solution}}$

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Say the LP is infeasible

$\boxed{\text{LP is unbounded.}}$

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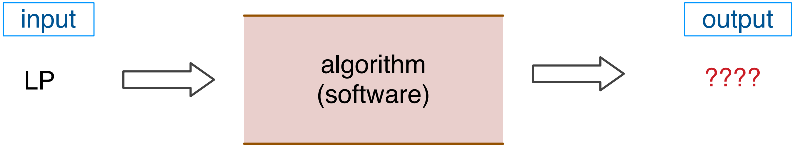

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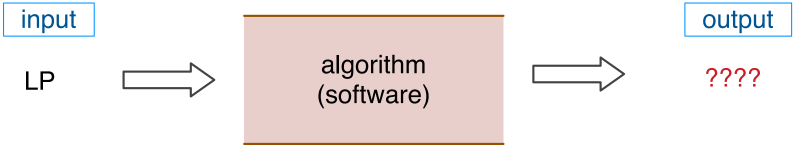


$\boxed{\text{LP has an optimal solution}}$

Return an optimal solution $\bar{x}$

$\boxed{\text{LP is infeasible.}}$

Say the LP is infeasible

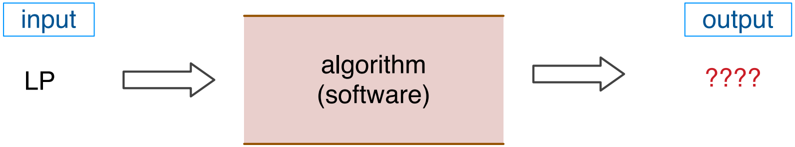
$\boxed{\text{LP is unbounded.}}$

Say the LP is unbounded

Remark

Algorithms should justify their answers !!!

We can now describe what we mean by solving a linear program,



$\boxed{\text{LP has an optimal solution}}$

Return an optimal solution $\bar{x}$ + proof that $\bar{x}$ is optimal.

$\boxed{\text{LP is infeasible.}}$

Return a proof the LP is infeasible.

$\boxed{\text{LP is unbounded.}}$

Return a proof the LP is unbounded.

Remark

Algorithms always need to justify their answers !!!

Recap

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1. Optimization problems can be:

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   (A) infeasible,

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   - indicating which of (A), (B), (C) holds,
   - if (C) holds give an optimal solution,
   - give a proof the answer is correct.

Module 2: Linear Programs (Certificates)

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

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Consider a linear program.

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Consider a linear program.

- If it is infeasible, how can we prove it?

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Questions

Consider a linear program.

- If it is infeasible, how can we prove it?
- If we have an optimal solution, how can we prove it is optimal?
- If it is unbounded, how can we prove it?

This can be always be done!

Proving Infeasibility

Proving Infeasibility

The following linear program is infeasible:

$$\max \quad (3, 4, -1, 2)^T x$$

s.t.

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Proving Infeasibility

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Question

How can we prove this problem is, in fact, infeasible?

Proving Infeasibility

The following linear program is infeasible:

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Question

How can we prove this problem is, in fact, infeasible?

We cannot try all possible assignments of values to x_1, x_2, x_3, and x_4.

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

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Proof

Construct a new equation:

$$\begin{array}{rrrrrrrrr} & -1 \times (1): & (-3 & 2 & 6 & -7)x & = & -6 & \\ + & 2 \times (2): & (4 & -2 & -4 & 8)x & = & 4 & \\ \hline & & (1 & 0 & 2 & 1)x & = & -2 & (\star) \end{array}$$

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Contradiction!

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

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$$(1 \;\; 0 \;\; 2 \;\; 1) x = -2 \qquad\qquad (\star)$$

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$$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} x = -2 \qquad (\star)$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

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Contradiction.

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b}$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

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Construct a new equation:

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$$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} x = -2 \qquad (\star) \qquad y^T A x = y^T b$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}. \qquad \underbrace{y^T A}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{y^T b}_{<0}$$

Contradiction.

This suggests the following result...

Proposition

There is no solution to $Ax = b,\ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

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Exercise

Give a proof of this proposition.

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Give a proof of this proposition.

Question

If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

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YES!!!!!

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If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

YES!!!!!

Farkas' Lemma

If there is no solution to $Ax = b,\ x \geq \mathbb{0}$, then there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Proving Optimality

$$\max \quad z(x) := (-1 \; -4 \;\; 0 \;\; 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

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Optimal solution:

$$\bar{x}_1 = 0$$

$$\bar{x}_2 = 0$$

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Question

How can we prove this solution is, in fact, optimal?

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Question

How can we prove this solution is, in fact, optimal?

We cannot try all possible feasible solutions.

$$\begin{array}{|ll|}\hline \max & z(x) := (-1\ -4\ \ 0\ \ 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \\ \hline \end{array}$$

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Claim

- $\bar{x}$ is feasible solution of value 4.

$$\begin{array}{ll} \max & z(x) := (-1 - 4 \;\; 0 \;\; 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

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Claim

- $\bar{x}$ is feasible solution of value 4. (easy)

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Let x' be an arbitrary feasible solution.

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Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1\ -4\ \ 0\ \ 0)}_{\leq \mathbb{0}} \underbrace{x'}_{\geq \mathbb{0}} + 4$$

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Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1 \; -4 \;\; 0 \;\; 0)x'}_{\leq 0} + 4 \leq 4.$$

Proving Unboudedness

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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Problem
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How can we prove that this problem is unbounded?

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Problem
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Question

How can we prove that this problem is unbounded?

Idea

Construct a family of feasible solutions $x(t)$ for all $t \geq 0$ and show that as t goes to infinity, the value of the objective function goes to infinity.

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

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$$x(t) := \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

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Claim 1

$x(t)$ is feasible for all $t \geq 0$.

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Claim 2

$z \to \infty$ when $t \to \infty$.

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$$Ax(t) =$$

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s.t.

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$z \to \infty$ when $t \to \infty$.

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Proof

$$z = c^T x(t)$$

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Proof

$$z = c^T x(t) = c^T[\bar{x} + tr] = c^T \bar{x} + t \underbrace{c^T r}_{=1>0} .$$

Exercise

Generalize and prove the following proposition.

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Generalize and prove the following proposition.

Proposition

The linear program,

$$\max\{c^T x : Ax = b, x \geq \mathbb{0}\}$$

is unbounded if we can find $\bar{x}$ and r such that

$$\bar{x} \geq \mathbb{0}, \quad r \geq \mathbb{0}, \quad A\bar{x} = b, \quad Ar = \mathbb{0} \quad \text{and} \quad c^T r > 0.$$

Recap

1. For linear programs, exactly one of the following holds. It is
   (A) infeasible,
   (B) unbounded, or
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1. For linear programs, exactly one of the following holds. It is
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3. If (B) occurs, there is a short proof of that fact.

4. For an optimal solution, there is a short proof that it is optimal.

Remark

We have not yet shown you how to find such proofs.

Module 2: Linear Programs (Standard Equality Forms)

The Key Definition

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A Linear Program (LP) is in Standard Equality Form (SEF) if

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1. it is a maximization problem,
2. for every variable x_j we have the constraint $x_j \geq 0$, and
3. all other constraints are equality constraints.

$$\max \quad (1,-2,4,-4,0,0)x + 3$$

s.t.

$$\begin{pmatrix} 1 & 5 & 3 & -3 & 0 & -1 \\ 2 & -1 & 2 & -2 & 1 & 0 \\ 1 & 2 & -1 & 1 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Question

Is the following LP in SEF?

$$
\begin{array}{ll}
\max & x_1 + x_2 + 17 \\
\text{s.t.} & \\
& x_1 - x_2 = 0 \\
& x_1 \geq 0
\end{array}
$$

Question

Is the following LP in SEF?

$$\boxed{\begin{array}{ll} \max & x_1 + x_2 + 17 \\ \text{s.t.} & \\ & x_1 - x_2 = 0 \\ & x_1 \geq 0 \end{array}}$$

NO! There is no constraint $x_2 \geq 0$.

Question

Is the following LP in SEF?

$$\boxed{\begin{array}{ll} \max & x_1 + x_2 + 17 \\ \text{s.t.} & \\ & x_1 - x_2 = 0 \\ & x_1 \geq 0 \end{array}}$$

NO! There is no constraint $x_2 \geq 0$. We say x_2 is free.

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Is the following LP in SEF?

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NO! There is no constraint $x_2 \geq 0$. We say x_2 is free.

Remarks

- $x_2 \geq 0$ is implied by the constraints.
- x_2 is still free since $x_2 \geq 0$ is not given explicitly.

Motivation

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We will develop an algorithm called the Simplex that can solve any LP

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What do we do if the LP is not in SEF?

Idea

1. Find an "equivalent" LP in SEF.

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Question

What do we do if the LP is not in SEF?

Idea

1. Find an "equivalent" LP in SEF.
2. Solve the "equivalent" LP using Simplex.

Motivation

We will develop an algorithm called the Simplex that can solve any LP

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Question

What do we do if the LP is not in SEF?

Idea

1. Find an "equivalent" LP in SEF.
2. Solve the "equivalent" LP using Simplex.
3. Use the sol'n of "equivalent" LP to get the sol'n of the original LP.

Motivation

We will develop an algorithm called the Simplex that can solve any LP

as long as it is in Standard Equality Form (SEF)

Question

What do we do if the LP is not in SEF?

Idea

1. Find an "equivalent" LP in SEF.
2. Solve the "equivalent" LP using Simplex.
3. Use the sol'n of "equivalent" LP to get the sol'n of the original LP.

Question

What do we mean by equivalent?

Equivalent LPs

Idea

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We will illustrate the proof with a series of examples.

Dealing with Minimization

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Free Variables

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$$\max \quad z = (1,2,3)(x_1,x_2,x_3)^\top$$

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$$\begin{pmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

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Any number is the difference between two non-negative numbers.

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$\boxed{\text{Set } x_3 := a - b \text{ where } a, b \geq 0.}$

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4. To solve any LP, it suffices to know how to solve LPs in SEF.

Module 2: Linear Programs (Simplex – A First Attempt)

A Naive Strategy for Solving an LP

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The SIMPLEX algorithm works along these lines.

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In this lecture: A first attempt at this algorithm.

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$$\boxed{\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

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Remarks

- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.

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Increase x_1 as much as possible, and keep x_2 unchanged, i.e.,

$$
\begin{aligned}
x_1 &= t \qquad \text{for some } t \geq 0 \text{ as large as possible} \\
x_2 &= 0
\end{aligned}
$$

$$\boxed{\begin{aligned} &\max \quad (4,3,0,0)x + 7 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &\qquad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}}$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

Choose $t \geq 0$ as large as possible.

$$\boxed{\begin{array}{ll} \max & (4,3,0,0)x+7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

Choose $t \geq 0$ as large as possible.

It needs to satisfy

1. the equality constraints, and

$$\boxed{\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

$$\begin{aligned} x_1 &= t \\ x_2 &= 0 \\ x_3 &= ? \\ x_4 &= ? \end{aligned}$$

Choose $t \geq 0$ as large as possible.

It needs to satisfy

1. the equality constraints, and
2. the non-negativity constraints.

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

Satisfying the Equality Constraints

$$\boxed{\begin{aligned} &\max \quad (4,3,0,0)x + 7 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &\qquad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}}$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x$$

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

$$\begin{aligned}
\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix}
\end{aligned}$$

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$
s.t.
$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\ &= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{aligned}$$

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

$$\begin{aligned}
\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}
\end{aligned}$$

$$\boxed{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

Satisfying the Equality Constraints

$$\boxed{\begin{aligned} \max \quad & (4,3,0,0)x + 7 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}}$$

$$\begin{aligned} x_1 &= t \\ x_2 &= 0 \\ x_3 &= ? \\ x_4 &= ? \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\ &= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{aligned}$$

$$\boxed{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

Remark

Equality constraints hold for any choice of t.

Satisfying the Non-Negativity Constraints

$$\begin{aligned} \max \quad & (4,3,0,0)x + 7 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$x_1 = t$$

$$x_2 = 0$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

Satisfying the Non-Negativity Constraints

$$\boxed{\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

$$x_1 = t$$
$$x_2 = 0$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

Satisfying the Non-Negativity Constraints

$$\begin{aligned} \max \quad & (4,3,0,0)x + 7 \\ \text{s.t.} \quad & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$x_1 = t$$
$$x_2 = 0$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

Satisfying the Non-Negativity Constraints

$$\begin{aligned} \max \quad & (4,3,0,0)x + 7 \\ \text{s.t.} \quad & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$x_1 = t$$
$$x_2 = 0$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0 \implies t \leq \frac{2}{3}$

Satisfying the Non-Negativity Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$

$$x_2 = 0$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0 \Rightarrow t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0 \Rightarrow t \leq 1$

Satisfying the Non-Negativity Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$

$$x_2 = 0$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0 \Rightarrow t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0 \Rightarrow t \leq 1$

Thus, the largest possible t is $\min\left\{1, \frac{2}{3}\right\} = \frac{2}{3}$.

Satisfying the Non-Negativity Constraints

$$\boxed{\begin{aligned} &\max \quad (4,3,0,0)x + 7 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &\qquad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}}$$

$$x_1 = t$$
$$x_2 = 0$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0 \Rightarrow t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0 \Rightarrow t \leq 1$

Thus, the largest possible t is $\min\left\{1, \frac{2}{3}\right\} = \frac{2}{3}$. The new solution is

$$x = (t,\ 0,\ 2 - 3t,\ 1 - t)^\top = \left(\frac{2}{3}, 0, 0, \frac{1}{3}\right)^\top$$

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_4 = \frac{1}{3}$$

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_4 = \frac{1}{3}$$

Question

Is the new solution optimal?

Repeating the Argument?

$$\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = \frac{2}{3}$$
$$x_2 = 0$$
$$x_3 = 0$$
$$x_4 = \frac{1}{3}$$

Question

Is the new solution optimal? NO!

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$
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Question

Is the new solution optimal? NO!

Question

Can we use the same trick to get a better solution?

Repeating the Argument?

$$\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = \frac{2}{3}$$
$$x_2 = 0$$
$$x_3 = 0$$
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Question

Is the new solution optimal? NO!

Question

Can we use the same trick to get a better solution? NO!

Repeating the Argument?

$$\begin{aligned} \max \quad & (4,3,0,0)x + 7 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$x_1 = \frac{2}{3} \qquad x_2 = 0 \qquad x_3 = 0 \qquad x_4 = \frac{1}{3}$$

Question

Is the new solution optimal? NO!

Question

Can we use the same trick to get a better solution? NO!

What made it work the first time around?

Remark

The LP needs to be in "canonical" form.

Remark

The LP needs to be in "canonical" form.

$$
\begin{array}{ll}
\max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix} x + 7 \\
\text{s.t.} & \\
& \begin{pmatrix}3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & (4 \quad 3 \quad 0 \quad 0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$x_1 = 0$

$x_2 = 0$

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Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & (4 \quad 3 \quad 0 \quad 0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = 0$$
$$x_2 = 0$$
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Revised strategy:

Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix} x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 2$$
$$x_4 = 1$$

Revised strategy:

Step 1. Find a feasible solution, x.

Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix}x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix}3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$x_1 = 0$

$x_2 = 0$

$x_3 = 2$

$x_4 = 1$

Revised strategy:

Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix} x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix}3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$x_1 = 0$

$x_2 = 0$

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Revised strategy:

Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Step 3. If x is optimal, STOP.

Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix}x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix}3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$x_1 = 0$

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Revised strategy:

Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Step 3. If x is optimal, STOP.

Step 4. If LP is unbounded, STOP.

Remark

The LP needs to be in "canonical" form.

$$
\boxed{
\begin{array}{ll}
\max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix} x + 7 \\
\text{s.t.} & \\
& \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
}
$$

$x_1 = 0$

$x_2 = 0$

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$x_4 = 1$

Revised strategy:

Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Step 3. If x is optimal, STOP.

Step 4. If LP is unbounded, STOP.

Step 5. Find a "better" feasible solution.

Remark

The LP needs to be in "canonical" form.

$$\boxed{\begin{array}{ll} \max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix}x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix}3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

$x_1 = 0$
$x_2 = 0$
$x_3 = 2$
$x_4 = 1$

Revised strategy:

Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Step 3. If x is optimal, STOP.

Step 4. If LP is unbounded, STOP.

Step 5. Find a "better" feasible solution.

From Here to a Complete Algorithm

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(1) Define what we mean by “canonical” form.

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algorithm known as the SIMPLEX.

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First on "To do list":

- Define basis and basic solutions.

From Here to a Complete Algorithm

(1) Define what we mean by "canonical" form.

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$\Longrightarrow$

algorithm known as the SIMPLEX.

First on "To do list":

- Define basis and basic solutions.
- Define canonical forms.

Module 2: Linear Programs (Basis)

Notation

Notation

Consider

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.

Notation

Consider

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{1, 2, 3\}$$

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{1, 2, 3\}$$

$$A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
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Notation

Consider

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{1, 3, 4\}$$

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{1, 3, 4\}$$

$$A_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Notation

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$$\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

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Consider

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{5\}$$

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{5\}$$

$$A_{\{5\}} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{5\} \qquad\qquad A_{\{5\}} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Notation

A_j denotes column j of A.

Notation

Consider

$$A = \begin{array}{c} \begin{matrix} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{matrix} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation

Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B.

$$B = \{5\} \qquad\qquad A_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Notation

A_j denotes column j of A.

Basis

Basis

Consider

$$\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ A = \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices.

Basis

Consider

$$\begin{array}{c@{}c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = & \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

Basis

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Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

Basis

Consider

$$\begin{array}{c@{}c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = & \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

Basis

Consider

$$\begin{array}{c} \\ A = \end{array} \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

Is $B = \{1, 2, 3\}$ a basis?

Basis

Consider

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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$$A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Basis

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

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Is $B = \{1, 2, 3\}$ a basis?

$$A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

YES

Basis

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

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Definition

Let B be a subset of column indices. B is a basis if

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(2) A_B is non-singular (columns are independent).

Is $B = \{1, 5\}$ a basis?

Basis

Consider

$$\begin{array}{c} \\ A = \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

Is $B = \{1, 5\}$ a basis?

$$A_B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

Basis

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

Is $B = \{1, 5\}$ a basis?

$$A_B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

NO

A_B is not square

Basis

Consider

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

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Let B be a subset of column indices. B is a basis if

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Is $B = \{2, 3, 4\}$ a basis?

Basis

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

Is $B = \{2, 3, 4\}$ a basis? $$A_B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Basis

Consider

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Definition

Let B be a subset of column indices. B is a basis if

(1) A_B is a square matrix,

(2) A_B is non-singular (columns are independent).

Is $B = \{2, 3, 4\}$ a basis?

$$A_B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

NO

columns of A_B are dependent

Question

Does every matrix have a basis?

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Does every matrix have a basis? NO.

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Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Theorem

Max number of independent columns $=$
Max number of independent rows.

Question

Does every matrix have a basis? NO.

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of A are dependent!
There are no 3 independent columns.

Theorem

Max number of independent columns $=$
Max number of independent rows.

Remark

Let A be a matrix with independent rows. Then B is a basis if and only if B is a maximal set of independent columns of A.

Basic Solutions

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{matrix} \color{red}1 & \color{red}2 & \color{red}3 & \color{red}4 & \color{red}5 \end{matrix} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,

Basic Solutions

$$\underbrace{\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \end{array}}_{}$$

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Example

Basis $B = \{1, 2, 4\}$.

Basic Solutions

$$\underbrace{\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \end{array}}_{}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Example

Basis $B = \{1, 2, 4\}$. Then

- x_1, x_2, x_4 are the basic variables,

Basic Solutions

$$\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

Let B be a basis of A.

- if $j \in B$ then x_j is a basic variable,
- if $j \notin B$ then x_j is a non-basic variable.

Example

Basis $B = \{1, 2, 4\}$. Then

- x_1, x_2, x_4 are the basic variables, and
- x_3, x_5 are the non-basic variables.

Basic Solutions

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Basic Solutions

$$\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

Basic Solutions

$$\underbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \multicolumn{5}{c}{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

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$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$: basic sol'n for $B = \{1, 2, 3\}$

Basic Solutions

$$\underbrace{\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \end{array}}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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(1) $Ax = b$

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}: \text{ basic sol'n for } B = \{1, 2, 3\}$$

(1) $Ax = b$ ✓

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$: basic sol'n for $B = \{1, 2, 3\}$

(1) $Ax = b$ ✓

(2) $x_4 = x_5 = 0$

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$: basic sol'n for $B = \{1, 2, 3\}$

(1) $Ax = b$ ✓

(2) $x_4 = x_5 = 0$ ✓

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Definition

x is a basic solution for basis B if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$: basic sol'n for $B = \{1, 3, 4\}$

Basic Solutions

$$\underbrace{\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}}_{A} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

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Thus, the basic solution is $x = (4,\ 0,\ 0,\ 2)^\top$.

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Did we have a choice for a basic solution x given $B = \{1, 4\}$?

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Question

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Basic Solutions – Uniqueness

Proposition

Consider $Ax = b$ and a basis B of A.
Then there exists a unique basic solution x for B.

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columns of A_B and elements of x_B are ordered by B!

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Since B is a basis, it implies A_B is non-singular, i.e., A_B^{-1} exists.

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Hence, $x_B = A_B^{-1} b$.

When Is a Vector Basic?

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Question

Is $x = (0, 0, 3, 0, 3)^\top$ basic?

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Proof

By contradiction. Suppose x is basic for basis B.

$$\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \underbrace{\begin{pmatrix} 3 & 2 & 1 & 4 & 1 \\ -1 & 1 & 0 & 2 & 1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}_{b}$$

Question

Is $x = (0, 1, 0, 1, 0)^\top$ basic? NO!

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is a column submatrix of A_B. But the columns of $A_{\{2,4\}}$ are dependent, so A_B is singular and B is not a basis, a contradiction.

Multiple Bases for a Basic Solution

$$\underbrace{\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \\ \multicolumn{5}{c}{\begin{pmatrix} 3 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix}} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{b}$$

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A basic solution can be the basic solution for more than one basis.

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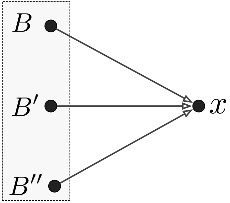

Relation to LPs

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Problem in SEF:

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We may assume, when solving (P), that rows of A are independent.

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Remark

We may assume, when solving (P), that rows of A are independent.

Definition

A basic solution x of $Ax = b$ is feasible if $x \geq 0$, i.e., if it is feasible for (P).

Consider the system

$$Ax = b$$

where the rows of A are independent.

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Recap

(1) B is a basis if A_B is a square, non-singular matrix.

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(6) A basic solution is feasible if it is non-negative.

Module 2: Linear Programs (Canonical Forms)

Consider

$$\max\left\{c^\top x : Ax = b, x \geq \mathbb{0}\right\} \tag{P}$$

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$$\begin{aligned}
\max \quad & (0 \quad 0 \quad 2 \quad 4)x \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Canonical form for $B = \{1, 2\}$

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$$\begin{array}{ll} \max & \begin{pmatrix} -2 & 0 & 0 & 6 \end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Canonical form for $B = \{2, 3\}$

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Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

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s.t.	s.t.
$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
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Example

$(1) \quad \bar{x} = (1, 2, 0, 0)^\top$ is feasible for both LPs.

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$$\boxed{\begin{array}{lcr} (0 \quad 0 \quad 2 \quad 4)\bar{x} & = & 2 \times 0 + 4 \times 0 = 0 \\ (-2 \quad 0 \quad 0 \quad 6)\bar{x} \quad +2 & = & -2 \times 1 + 6 \times 0 + 2 = 0 \end{array}}$$

Illustration with an Example

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$$\boxed{\begin{array}{ll} \max & \underbrace{\begin{pmatrix}0 & 0 & 2 & 4\end{pmatrix}}_{c} x \\ \text{s.t.} & \\ & \underbrace{\begin{pmatrix}1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1 \\ 2\end{pmatrix}}_{b} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}} \qquad \text{(P)}$$

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Question

How do we rewrite (P) in canonical form for basis $B = \{2, 3\}$?

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(P2) Replace $c^\top x$ by $\bar{c}^\top x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and $\bar{z}$ is a constant.

Rewriting Constraints – Example

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Remarks

- $A'_B = I$.

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$$Ax = b$$

$\Longleftrightarrow$

$$\underbrace{A_B^{-1} A}_{A'} x = \underbrace{A_B^{-1} b}_{b'}$$

Remarks

- $A'_B = I$.
- $Ax = b$ and $A'x = b'$ have the same set of solutions.

Rewriting the Objective Function – Example

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$$\max \quad z = \underbrace{\begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix}}_{c^\top} x$$

s.t.

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$$
\boxed{
\begin{aligned}
&\max \quad z = (0 \quad 0 \quad 2 \quad 4)x \\
&\text{s.t.} \\
&\qquad \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
&\qquad x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
}
$$

$$
(y_1, y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = (y_1, y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}
$$

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\hline
\end{array}
$$

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Remark

For any choice of y_1, y_2 and any feasible solution x,

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For any choice of y_1, y_2 and any feasible solution x,

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Question

How do we choose y_1, y_2 such that $\bar{c}_2 = \bar{c}_3 = 0$?

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$\Longleftrightarrow$

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\ 2\end{pmatrix} = \begin{pmatrix}2\\ 0\end{pmatrix}$$

$$z = \underbrace{\left[\begin{pmatrix}0 & 0 & 2 & 4\end{pmatrix} - \begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2\end{pmatrix}\right]}_{\bar{c}^\top} x + \underbrace{\begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}1 \\ 2\end{pmatrix}}_{\bar{z}}$$

Question

How do we choose y_1, y_2 such that $\bar{c}_B = 0$ for $B = \{2, 3\}$?

Choose

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⇒

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$\Rightarrow$

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$\Rightarrow$

$$z = \begin{pmatrix}-2 & 0 & 0 & 6\end{pmatrix} x + 2$$

Rewriting the Objective Function – General

$$\boxed{\begin{aligned} \max \quad & z = c^\top x \\ \text{s.t.} \quad & \\ & Ax = b \\ & x \geq \mathbb{0} \end{aligned}}$$

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$$y^\top A x = y^\top b$$

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Remark

For any choice of y and any feasible solution x,

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How do we choose y such that $\bar{c}_B = 0$ for a basis B?

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$$\mathbb{0}^\top = \bar{c}_B^\top = c_B^\top - y^\top A_B$$

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Remark

For any non-singular matrix M,

$$(M^\top)^{-1} = (M^{-1})^\top =: M^{-\top}$$

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Recap

Proposition

Let B be a basis of A,

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbb{0} \end{array}} \quad \text{(P)}$$

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$$\boxed{\begin{aligned} \max \quad & \underbrace{\left[c^\top - y^\top A\right]}_{\bar{c}} x + y^\top b \\ \text{s.t.} \quad & \\ & \underbrace{A_B^{-1} A}_{A'} x = A_B^{-1} b \\ & x \geq \mathbb{0} \end{aligned}} \quad \text{(P')}$$

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where $y = A_B^{-\top} c_B$.

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(1) (P') is in canonical form for basis B, i.e., $\bar{c}_B = 0$ and $A'_B = I$.

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where $y = A_B^{-\top} c_B$. Then

(1) (P') is in canonical form for basis B, i.e., $\bar{c}_B = 0$ and $A'_B = I$.

(2) (P) and (P') have the same feasible region.

(3) Feasible solutions have the same objective value for (P) and (P').

Module 2: Linear Programs (Formalizing the Simplex)

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular

Finding an Optimal Solution

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Rightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$ $\Rightarrow$ LP is in canonical form for B

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

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Finding an Optimal Solution

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- $\bar{x} \geq \mathbf{0}$

Finding an Optimal Solution

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Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$ $\Longrightarrow$ LP is in canonical form for B
- $\bar{x} = (2, 0, 0, 5)^\top$ is a the basic solution for B.
- $\bar{x} \geq \mathbf{0}$ $\Longrightarrow$ $\bar{x}$ is feasible, i.e., B is feasible

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

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Canonical form for B

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Question

How do we find a better feasible solution?

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

Set $x_k = t \geq 0$ as large as possible.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

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$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

Set $x_k = t \geq 0$ as large as possible.

Keep all other non-basic variables at 0.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

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Idea

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Keep all other non-basic variables at 0.

Pick $k = 2$. Set $x_2 = t \geq 0$.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

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$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\max \quad \underbrace{\begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 1 & 2 & 0\\ 0 & 1 & 1 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\ 5\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{pmatrix}2\\ 5\end{pmatrix} = \begin{pmatrix}1 & 1 & 2 & 0\\ 0 & 1 & 1 & 1\end{pmatrix} x$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned} \begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned}
\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
\max \quad & \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x \\
\text{s.t.} \quad & \\
& \underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$
\begin{aligned}
\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}
\end{aligned}
$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0,\ x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned}
\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}
\end{aligned}$$

$$\boxed{\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

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$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Rightarrow \quad t \leq \frac{2}{1}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Longrightarrow \quad t \leq \tfrac{2}{1}$$

$$x_4 = 5 - t \geq 0 \quad \Longrightarrow \quad t \leq \tfrac{5}{1}$$

$$\max \quad \underbrace{\begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 1 & 2 & 0\\ 0 & 1 & 1 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\5\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix}x_1\\x_4\end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix}2\\5\end{pmatrix}}_{b} - t\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{2}{1}$$

$$x_4 = 5 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{5}{1}$$

Thus, the largest possible $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\}$.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{2}{1}$$

$$x_4 = 5 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{5}{1}$$

Thus, the largest possible $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\}$.

The new feasible solution is $x = (0, 2, 0, 3)^\top$. It has value $2 > 0$.

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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Question

For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

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For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$x_2 \neq 0 \quad \Rightarrow \quad 2 \in B$

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix}1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 5\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

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For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$x_2 \neq 0 \Rightarrow 2 \in B$

$x_4 \neq 0 \Rightarrow 4 \in B$

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix}1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 5\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

Question

For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$$x_2 \neq 0 \Rightarrow 2 \in B$$
$$x_4 \neq 0 \Rightarrow 4 \in B$$

As $|B| = 2$, $B = \{2, 4\}$.

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

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OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

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$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$\Downarrow$

$$
\begin{aligned}
&\max \quad (0 \quad 1 \quad 3 \quad 0)x \\
&\text{s.t.} \\
&\qquad \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
&\qquad x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\begin{array}{ll} \max & \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

$$\max \quad (0 \quad 1 \quad 3 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\Downarrow$$

$$\max \quad (-1 \quad 0 \quad 1 \quad 0)x + 2$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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$\{1, 4\}$ is a feasible basis

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$$\begin{array}{ll} \max & \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

- 2 entered the basis.

$$\begin{aligned} \max \quad & (0 \quad 1 \quad 3 \quad 0)x \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\begin{aligned} \max \quad & (-1 \quad 0 \quad 1 \quad 0)x + 2 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

- 2 entered the basis.
- 1 left the basis.

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1,4\}$ is a feasible basis

Canonical form for $\{1,4\}$

$$\begin{array}{ll} \max & \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

NEW

$\{2,4\}$ is a feasible basis

Canonical form for $\{2,4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

- 2 entered the basis.
- 1 left the basis.

WHY?

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

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OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

$\Longrightarrow$ 2 enters the basis

$$\begin{array}{ll} \max & (0 \quad 1 \quad 3 \quad 0)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

➡ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

➡ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\} = 2$.

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

➡ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\} = 2$.

➡ $x_1 = 0$ and 1 leaves the basis

$$\max \quad (0 \quad 1 \quad 3 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

➡ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $t = \min \left\{ \frac{2}{1}, \frac{5}{1} \right\} = 2$.

➡ $x_1 = 0$ and 1 leaves the basis

The NEW basis is $\{2, 4\}$.

Example – Continued

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix}-1 & 0 & 1 & 0\end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 1 & 2 & 0\\ -1 & 0 & -1 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\ 3\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$$x_3 = t$$

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix}-1 & 0 & 1 & 0\end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 1 & 2 & 0\\ -1 & 0 & -1 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\3\end{pmatrix}}_{b}$$

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$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ⟹ 3 enters the basis

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ $\Longrightarrow$ 3 enters the basis

Pick $x_B = b - tA_k$:

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ $\Longrightarrow$ 3 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix}-1 & 0 & 1 & 0\end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 1 & 2 & 0\\ -1 & 0 & -1 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\3\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ➡ 3 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix}x_2\\x_4\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} - t\begin{pmatrix}2\\-1\end{pmatrix}$$

$t = \min\left\{\frac{2}{2}, -\right\} = 2$ thus $x_2 = 0$

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ➡ 3 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$t = \min\left\{\frac{2}{2}, -\right\} = 2$ thus $x_2 = 0$ ➡ 2 leaves the basis

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix}-1 & 0 & 1 & 0\end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 1 & 2 & 0\\ -1 & 0 & -1 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\ 3\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t \quad \Rightarrow \quad 3$ enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix}x_2\\ x_4\end{pmatrix} = \begin{pmatrix}2\\ 3\end{pmatrix} - t\begin{pmatrix}2\\ -1\end{pmatrix}$$

$t = \min\left\{\frac{2}{2}, -\right\} = 2$ thus $x_2 = 0 \quad \Rightarrow \quad 2$ leaves the basis

The NEW basis is $B = \{3, 4\}$.

$$\max \quad \underbrace{\begin{pmatrix} -1.5 & -0.5 & 0 & 0 \end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix} 0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix}0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1 \\ 4\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix}0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1 \\ 4\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix}0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1 \\ 4\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix}0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1 \\ 4\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix}0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1 \\ 4\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: ???

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

Let x be a feasible solution.

$$\max \quad \underbrace{(-1.5 \quad -0.5 \quad 0 \quad 0)}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix} 0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

Let x be a feasible solution. Then

$$\underbrace{(-1.5,\ 0.5,\ 0,\ 0)}_{\leq \mathbf{0}} \underbrace{x}_{\geq \mathbf{0}} + 3$$

$$\max \quad \underbrace{(-1.5 \quad -0.5 \quad 0 \quad 0)}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix} 0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

Let x be a feasible solution. Then

$$\underbrace{(-1.5,\ 0.5,\ 0,\ 0)}_{\leq \mathbf{0}} \underbrace{x}_{\geq \mathbf{0}} + 3 \leq 3.$$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 4, 5\}$ is a feasible basis

Canonical form for $\{1, 4, 5\}$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 4, 5\}$ is a feasible basis

Canonical form for $\{1, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

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The NEW basis is $B = \{3, 4, 5\}$.

$$\max \quad \begin{pmatrix} -3 & 2 & 0 & 0 & 0 \end{pmatrix} x + 3$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

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Canonical form for $\{3, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_2 = t$ $\Longrightarrow$ 2 enters the basis

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s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

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$\{3, 4, 5\}$ is a feasible basis

Canonical form for $\{3, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_2 = t$ $\Rightarrow$ 2 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} - t \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \max \quad & \begin{pmatrix} -3 & 2 & 0 & 0 & 0 \end{pmatrix} x + 3 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

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Choose $t =???$

$$
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Claim

The linear program is unbounded.

$$\max \quad z = (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3$$

s.t.

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$$x(t) = \begin{pmatrix} 0 \\ t \\ 1 + 2t \\ 4 + t \\ 4 \end{pmatrix} =$$

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- $x(t)$ is feasible for all $t \geq 0$.
- $z \to \infty$ when $t \to \infty$.

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($\bar{x}, r$: certificate of unboundedness.)

The Simplex Algorithm

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OUTPUT:

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Step 1. Rewrite in canonical form for the basis B.

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Step 2. Find a better basis B or get required outcome.

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Trying to Find a Better Basis

$$\boxed{\begin{aligned} &\max \quad z = c_N^\top x_N + \bar{z} \\ &\text{s.t.} \\ &\qquad\quad x_B + A_N x_N = b \\ &\qquad\quad x \geq \mathbf{0} \end{aligned}}$$

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B is a feasible basis, $N = \{j \notin B\}$

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Canonical form for B

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$\bar{x}$ is a basic solution

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If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$.

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If $A_k \leq \mathbf{0}$, then STOP. The LP is unbounded.

Choose $t = \min\left\{\frac{b_i}{A_{ik}} : \text{for all } i \text{ such that } A_{ik} > 0\right\}$.

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$\bar{x}$ is a basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$.

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If $A_k \leq \mathbf{0}$, then STOP. The LP is unbounded.

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Let x_r be a basic variable forced to 0.

Trying to Find a Better Basis

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The new basis is obtained by having k enter and r leave.

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Let us see an example...

$$\max \quad (0 \quad 0 \quad 2 \quad 3)x$$

s.t.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 2\}$ is a feasible basis

Canonical form for $\{1, 2\}$

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Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} - t \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \text{and} \quad t = \min\left\{\tfrac{6}{2}, \tfrac{12}{4}\right\} = 3$$

Pick $r \in B$ such that $x_r = 0$:

Choices $r = 1$

$$\max \quad \begin{pmatrix} 0 & 0 & 2 & 3 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 2\}$ is a feasible basis

Canonical form for $\{1, 2\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

Choices $k = 3$ OR $k = 4$.

Bland's rule says pick $k = 3$ (entering element).

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} - t \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \text{and} \quad t = \min\left\{\tfrac{6}{2}, \tfrac{12}{4}\right\} = 3$$

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$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 2 & 3 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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Bland's rule says pick $r = 1$ (leaving element).

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s.t.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

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The NEW basis is $B = \{3, 4\}$.

Recap

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- We have seen a formal description of the Simplex algorithm.
- We showed that if the algorithm terminates, then it is correct.
- We defined Bland's rule and asserted, without proof, that Simplex terminates as long as we are using Bland's rule.
- To get started, we need to get a feasible basis.

  To do: Find a procedure to find a feasible basis.

Module 2: Linear Programs (Finding a Feasible Solution)

The Problem

Consider

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}.$$

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How do we find a feasible solution?

These two questions are equivalent.

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$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}.$$

To run Simplex, we need a feasible basis.

Question

How do we find a feasible basis?

An easier question,

Question

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These two questions are equivalent.

Exercise

There is an algorithm that, given a feasible solution, finds a feasible basis.

The Problem

Consider

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}.$$

To run Simplex, we need a feasible basis.

Question

How do we find a feasible basis?

An easier question,

Question

How do we find a feasible solution?

These two questions are equivalent.

Exercise

There is an algorithm that, given a feasible solution, finds a feasible basis.

$\Longrightarrow$ We will focus on the second question.

The Key Idea

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Algorithm 1

INPUT: A, b, c, and a feasible solution

OUTPUT: Optimal solution/detect LP unbounded.

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Algorithm 1

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OK
Simplex +
exercise

The Key Idea

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}$$

Algorithm 1
<u>INPUT:</u> A, b, c, and a feasible solution
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OK
Simplex +
exercise

Algorithm 2
<u>INPUT:</u> A, b, c.
<u>OUTPUT:</u> Feasible solution/detect there is none.

The Key Idea

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Algorithm 1

<u>INPUT:</u> A, b, c, and a feasible solution

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<u>INPUT:</u> A, b, c.

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HOW?

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HOW?

We will show that...

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OK
Simplex +
exercise

Algorithm 2	
INPUT:	A, b, c.
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HOW?

We will show that...

Proposition

We can use Algorithm 1 to get Algorithm 2.

A First Example

A First Example

Problem: Find a feasible solution/detect none exist for

$$\begin{array}{ll} \max & (1,2,-1,3)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

A First Example

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$$\begin{array}{ll} \max & (1,2,-1,3)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Remark

It does not depend on the objective function.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.

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Step 1. Multiply the equations such that the RHS is non-negative. OK

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$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\min \quad x_5 + x_6$$

s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

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$$\boxed{\begin{aligned} &\min \quad x_5 + x_6 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}}$$

x_5, x_6 are the auxiliary variables

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Remark

The auxiliary problem is

<u>Problem:</u> Find a feasible solution/detect none exist for

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

min $x_5 + x_6$

s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$x \geq \mathbf{0}$

x_5, x_6 are the auxiliary variables

Remark

The auxiliary problem is

- feasible, since $(0,0,0,0,7,13)^\top$ is a solution, and

Problem: Find a feasible solution/detect none exist for

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned} \min \quad & x_5 + x_6 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

x_5, x_6 are the auxiliary variables

Remark

The auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and
- bounded, as 0 is the lower bound.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned} \min \quad & x_5 + x_6 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

x_5, x_6 are the auxiliary variables

Remark

The auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and
- bounded, as 0 is the lower bound.

Therefore, the auxiliary problem has an optimal solution.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \tag{$\star$}$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

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x_5, x_6 are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

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Step 3. Solve the auxiliary problem using Algorithm 1.

$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

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since $x_5 = x_6 = 0$.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{ll} \min & x_5 + x_6 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

x_5, x_6 are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

since $x_5 = x_6 = 0$.

Therefore, $(2, 1, 0, 0)^\top$ is a feasible solution to $(\star)$.

A Second Example

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \tag{$\star$}$$

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A Second Example

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$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\boxed{\begin{array}{l} \min \quad z = x_4 + x_5 \\ \text{s.t.} \\ \qquad \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}}$$

x_4, x_5 are the auxiliary variables

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

x_4, x_5 are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned} &\min \quad z = x_4 + x_5 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$

x_4, x_5 are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(0, 0, 1, 0, 3)^\top$ is an optimal solution to the auxiliary problem.

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\boxed{\begin{aligned} &\min \quad z = x_4 + x_5 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}}$$

x_4, x_5 are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(0, 0, 1, 0, 3)^\top$ is an optimal solution to the auxiliary problem.

However, $(0, 0, 1)^\top$ is NOT a solution to $(\star)$.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

$$\boxed{\begin{array}{ll} \min & z = x_4 + x_5 \\ \text{s.t.} & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

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the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$\min \quad z = x_4 + x_5$ s.t. $\begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ $x \geq \mathbf{0}$

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Claim

$(\star)$ does not have a solution.

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$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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$(\star)$ does not have a solution.

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Suppose, for a contradiction, $(\star)$ has a solution x'_1, x'_2, x'_3.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x'_1, x'_2, x'_3.

Then, $(x'_1, x'_2, x'_3, 0, 0)^\top$ is a feasible solution to the auxiliary problem,

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Suppose, for a contradiction, $(\star)$ has a solution x'_1, x'_2, x'_3.

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Formalize

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Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Step 1. Multiply the equations such that b is non-negative.

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Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

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$$\boxed{\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \begin{pmatrix} A & \mid & I \end{pmatrix} x = b \\ & x \geq \mathbf{0} \end{array}}$$

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$x_{n+1}, \ldots, x_{n+m}$ are the auxiliary variables

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Step 3. Solve the auxiliary problem using Algorithm 1.

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Proposition

If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to $(\star)$.

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Proposition

If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to ($\star$).

Proof

When $z = 0$, we must have $x_{n+1} = \ldots = x_{n+m} = 0$.

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Proposition

When $z > 0$, then $(\star)$ has no solution.

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

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When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

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Suppose, for a contradiction, ($\star$) has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

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The 2-Phase Method

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Phase 2. Given that feasible solution:
find an optimal solution/detect LP unbounded.

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Example

Solve the following LP,

$$\boxed{\begin{array}{ll} \max & (1,1,1)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

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$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

Phase 1. Find a feasible solution/detect none exist for

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Step 1. Multiply the equations such that the RHS is non-negative. OK

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$$\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

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NOT in SEF

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = -x_4 - x_5$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

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In SEF

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In SEF

feasible basis $B = \{4, 5\}$

Phase 1. Find a feasible solution/detect none exist for

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In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

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In SEF

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To rewrite $B = \{4, 5\}$ in canonical form, you can

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To rewrite $B = \{4, 5\}$ in canonical form, you can

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In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

To rewrite $B = \{4, 5\}$ in canonical form, you can

- use the formulae, OR
- notice $A_B = I$ and rewrite the objective function as follows...

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

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In SEF

feasible basis $B = \{4, 5\}$

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$$z = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \end{pmatrix} x$$

Phase 1. Find a feasible solution/detect none exist for

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feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{aligned} z &= (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\ 0 &= (1 \quad 2 \quad -1 \quad 1 \quad 0)x - 4 \end{aligned}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{ll} \max & z = -x_4 - x_5 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{array}{rcllllllll} z & = & (0 & 0 & 0 & -1 & -1)x & & \\ 0 & = & (1 & 2 & -1 & 1 & 0)x & - & 4 \\ 0 & = & (1 & -1 & 1 & 0 & 1)x & - & 4 \end{array}$$

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s.t.

$$\begin{pmatrix}1 & 2 & -1 & 1 & 0\\ 1 & -1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}4\\4\end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{array}{rcllllll}
z & = & (0 & 0 & 0 & -1 & -1)x & \\
0 & = & (1 & 2 & -1 & 1 & 0)x & - \quad 4\\
0 & = & (1 & -1 & 1 & 0 & 1)x & - \quad 4\\
\hline
z & = & (2 & 1 & 0 & 0 & 0)x & - \quad 8
\end{array}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

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In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

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In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

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$$\begin{pmatrix}1 & 2 & -1\\ 1 & -1 & 1\end{pmatrix} x = \begin{pmatrix}4\\4\end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = (2 \quad 1 \quad 0 \quad 0 \quad 0) - 8$$

s.t.

$$\begin{pmatrix}1 & 2 & -1 & 1 & 0\\ 1 & -1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}4\\4\end{pmatrix}$$

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In SEF

feasible basis $B = \{4, 5\}$

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Step 3. Solve the auxiliary problem using Simplex, starting from B.

$B = \{1, 4\}$ is an optimal basis with the basic solution $(4, 0, 0, 0, 0)^\top$.

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In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

$B = \{1, 4\}$ is an optimal basis with the basic solution $(4, 0, 0, 0, 0)^\top$.

$z = 0$ implies that $(4, 0, 0)^\top$ is a feasible solution for $(\star)$.

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

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Remark

$(4, 0, 0)^\top$ is a basic solution.

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Remark

$(4,0,0)^\top$ is a basic solution.

Exercise

Show that this will always be the case!

Phase 1. $(4,0,0)^\top$ is a feasible solution for

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Question

For what basis, B, is $x = (4,0,0)^\top$ a basic solution?

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$x_1 \neq 0$

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Question

For what basis, B, is $x = (4,0,0)^\top$ a basic solution?

$x_1 \neq 0 \Longrightarrow 1 \in B$

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For what basis, B, is $x = (4,0,0)^\top$ a basic solution?

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Cardinality of maximal set of independent columns of $A =$
Cardinality of maximal set of independent rows of $A = 2$

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Thus, for some $i \in \{2,3\}$, columns 1 and i of A are independent.

In this case, we can pick $i = 2$. In particular, $B = \{1,2\}$ is a basis.

Phase 2. Find an optimal solution/detect LP unbounded.

$$\begin{array}{ll} \max & (1,1,1)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

$B = \{1, 2\}$ is a feasible basis (from Phase 1).

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We can now solve the problem using Simplex, starting from B.

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$B = \{1, 2\}$ is a feasible basis (from Phase 1).

We can now solve the problem using Simplex, starting from B.

$\Rightarrow$

$x = (0, 8, 12)^\top$ is an optimal solution.

Consequences

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Theorem

$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}$$

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Consequences

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Proof

Run 2-Phase method with Simplex using Bland's rule.

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Proof

Run 2-Phase method with Simplex using Bland's rule.
(Recall that Bland's rule ensures that Simplex terminates.)

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A finite number of basis implies a finite number of basic solutions.

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If a LP has at least two feasible solutions and one optimal solution,

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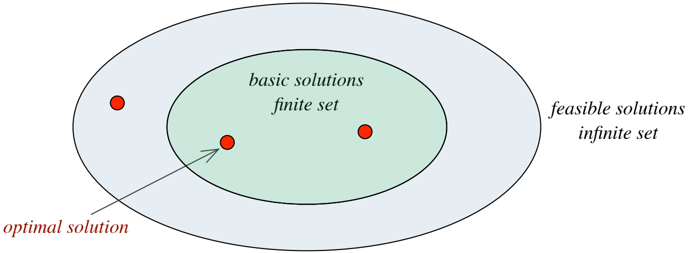

Fundamental Theorem of Linear Programming

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Consider an arbitrary LP.

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Proof

Convert the LP into an equivalent LP in SEF.

Fundamental Theorem of Linear Programming

Consider an arbitrary LP. Exactly one of the following holds:

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Proof

Convert the LP into an equivalent LP in SEF.

Apply the previous theorem.

Recap

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- Using 2-Phase + the Simplex algorithm, we can solve arbitrary LPs.

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Sergio Schnitzler/Hemera/Thinkstock

State of the art implementation

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State of the art implementation

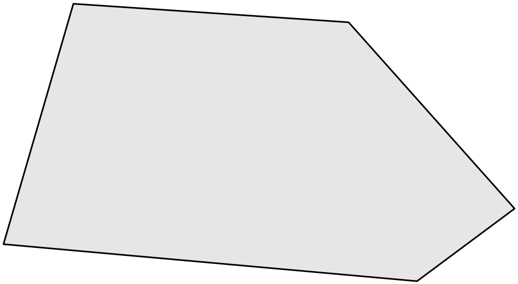
gamegfx/iStock/Thinkstock

Our implementation

Module 2: Linear Programs (Extreme Points)

Extreme Points

Consider the following convex set:



Extreme Points

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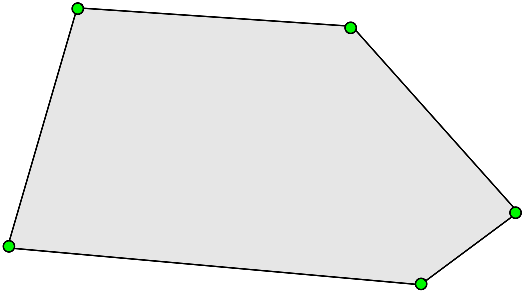

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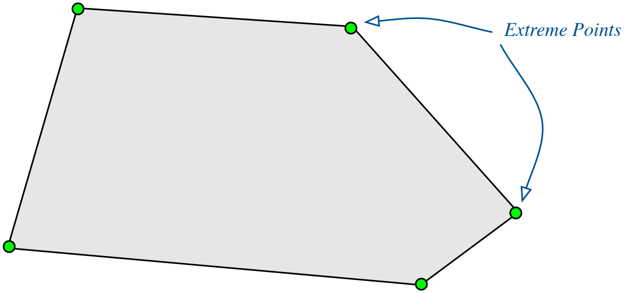

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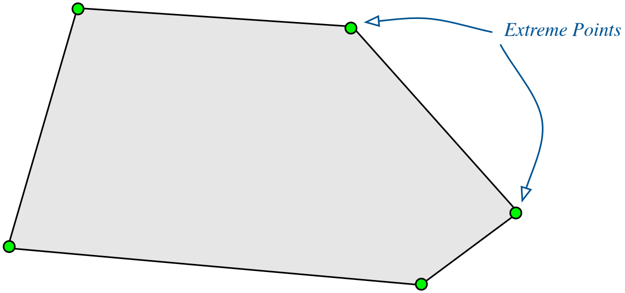

Question

How might we formally describe the “extreme points”?

Towards a Definition of Extreme Points

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Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

Towards a Definition of Extreme Points

Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

- $x \in L$ and

Towards a Definition of Extreme Points

Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

- $x \in L$ and
- x is distinct from the endpoints of L.

Towards a Definition of Extreme Points

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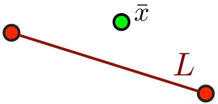


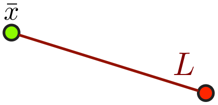
$\bar{x}$ is not contained in L.

Towards a Definition of Extreme Points

Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

- $x \in L$ and
- x is distinct from the endpoints of L.



$\bar{x}$ is contained in L,

but NOT properly.

Towards a Definition of Extreme Points

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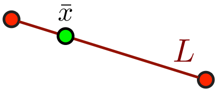

$\bar{x}$ is PROPERLY contained in L.

Towards a Definition of Extreme Points

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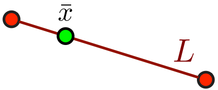

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Definition

Let S be a convex set and $\bar{x} \in S$.

Towards a Definition of Extreme Points

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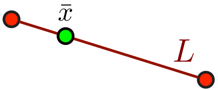

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Definition

Let S be a convex set and $\bar{x} \in S$. Then $\bar{x}$ is NOT an extreme point if

Towards a Definition of Extreme Points

Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

- $x \in L$ and
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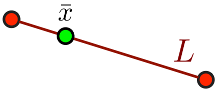

$\bar{x}$ is PROPERLY contained in L.

Definition

Let S be a convex set and $\bar{x} \in S$. Then $\bar{x}$ is NOT an extreme point if there exists a line segment $L \subseteq S$ where L properly contains $\bar{x}$.

Extreme Points - Examples

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Let S be a convex set and $\bar{x} \in S$. Then $\bar{x}$ is NOT an extreme point if there exists a line segment $L \subseteq S$ where L properly contains $\bar{x}$.

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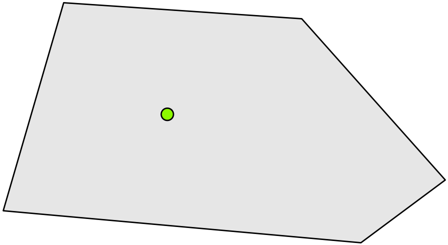

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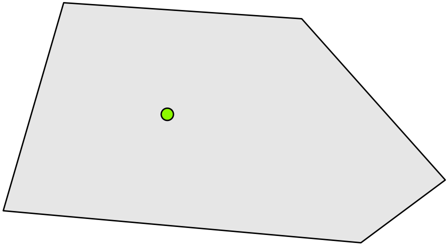

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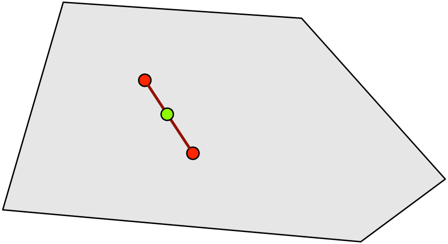

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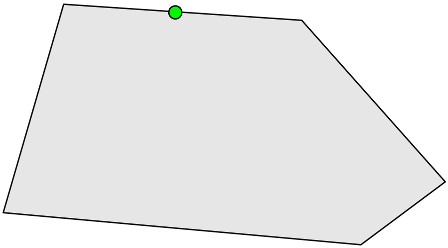

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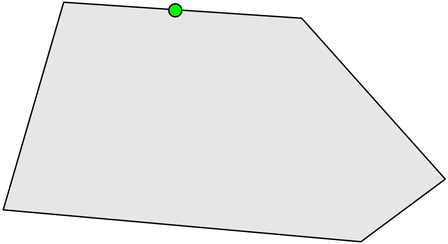

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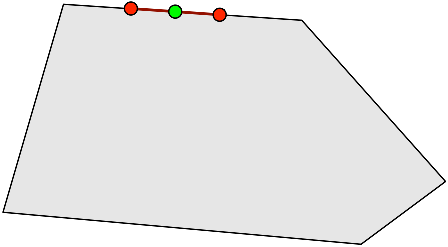

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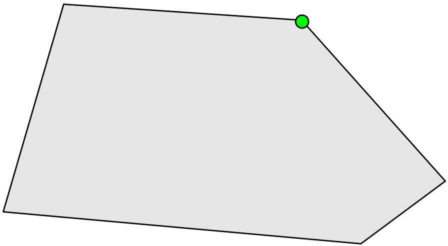

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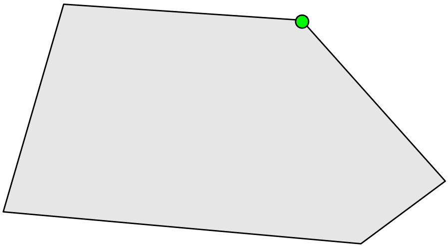

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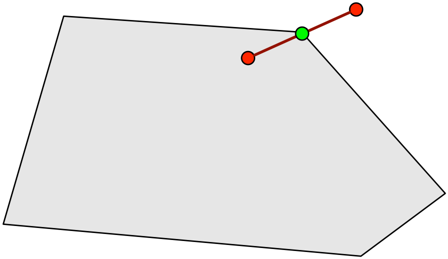

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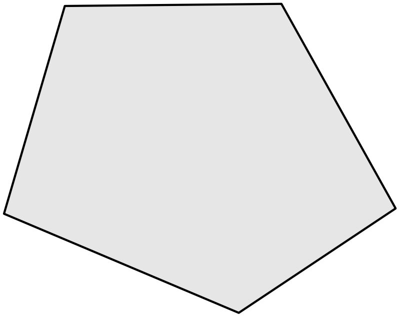

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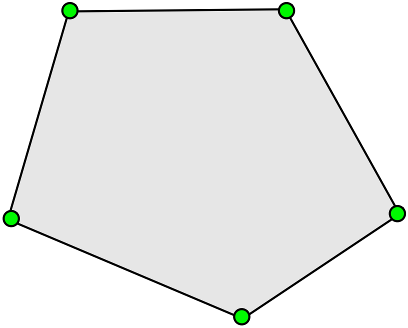

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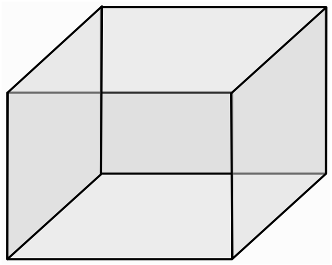

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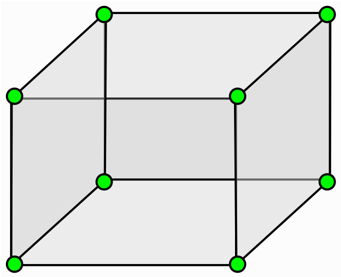

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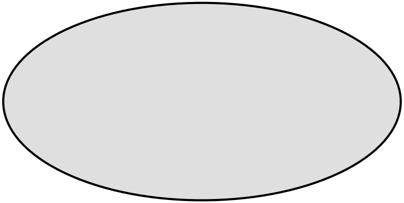

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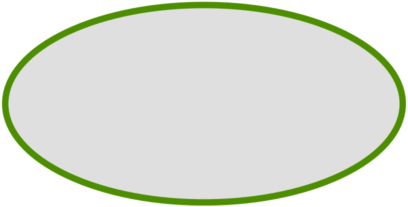

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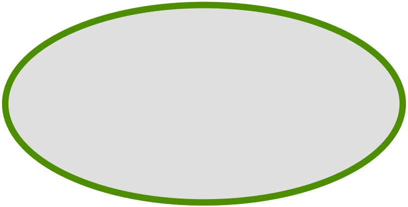

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A convex set may have an infinite number of extreme points.

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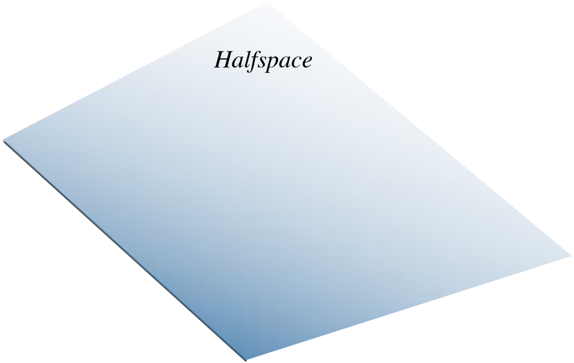

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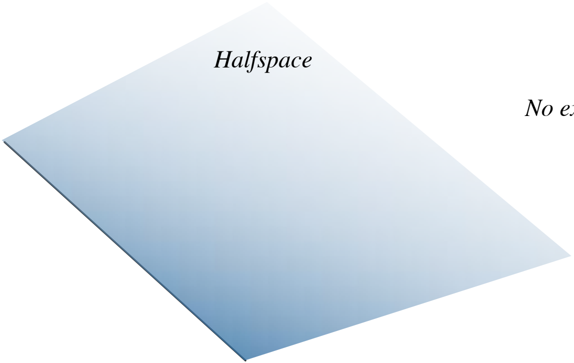

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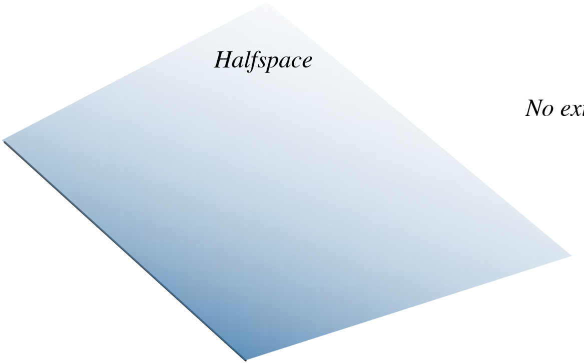

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1. Characterize the extreme points in a polyhedron.
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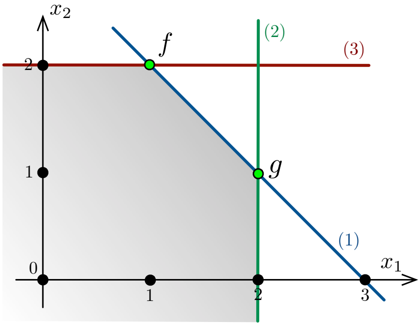
3. Gain a geometric understanding of the Simplex algorithm.



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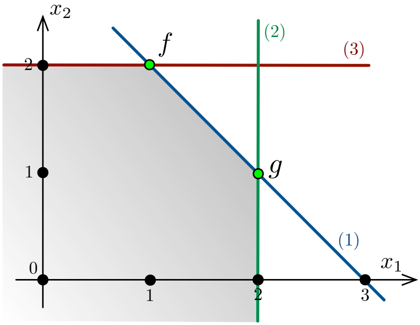

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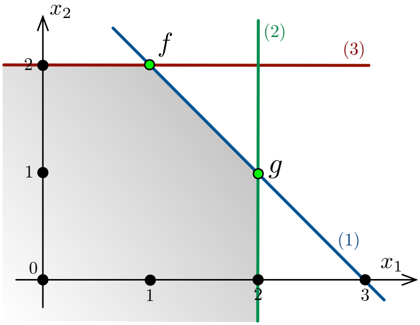

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Each satisfy $n = 2$ "independent" constraints with equality!

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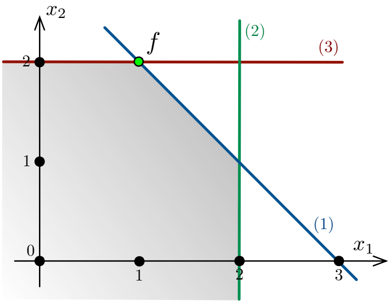

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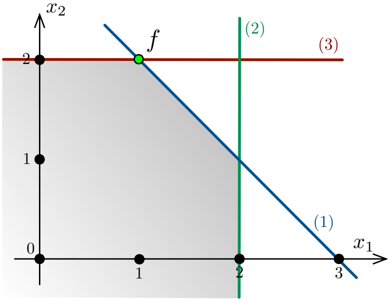

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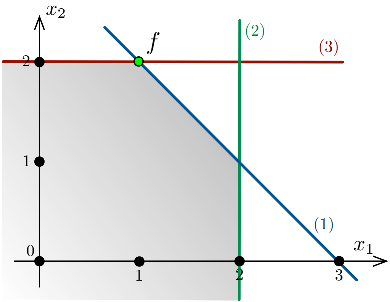

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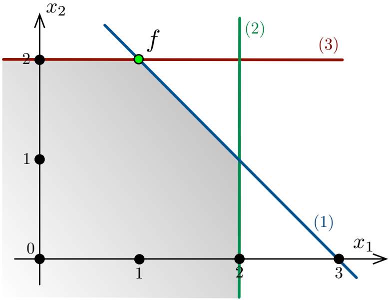

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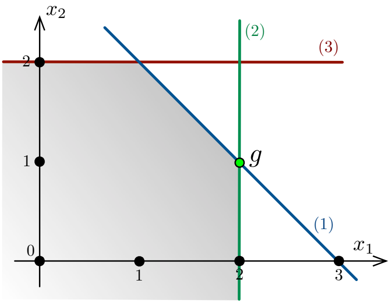

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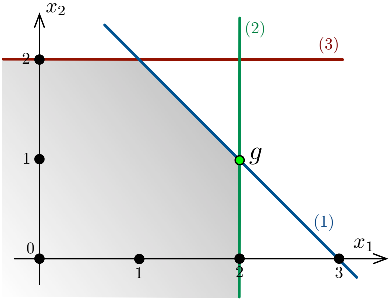

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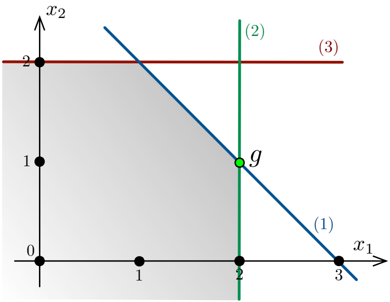

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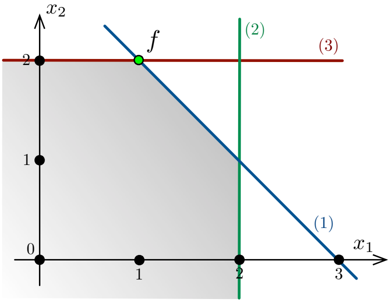

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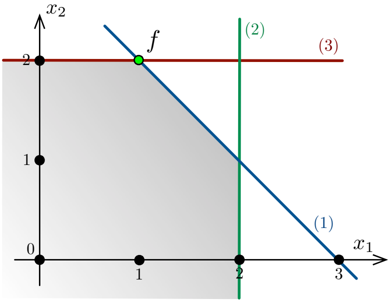

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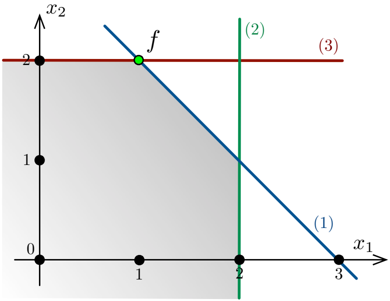

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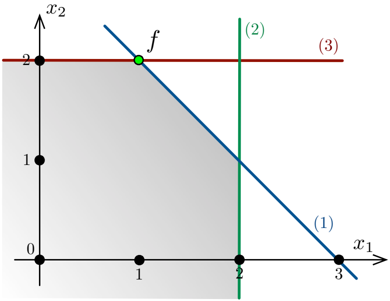

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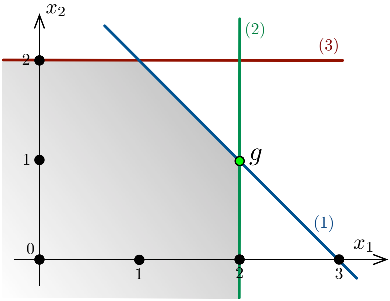

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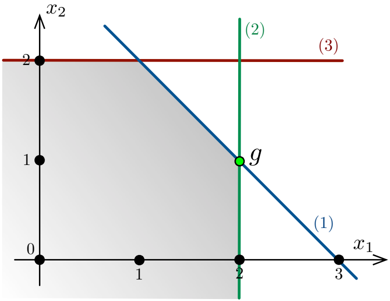

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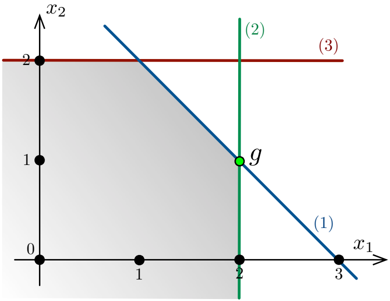

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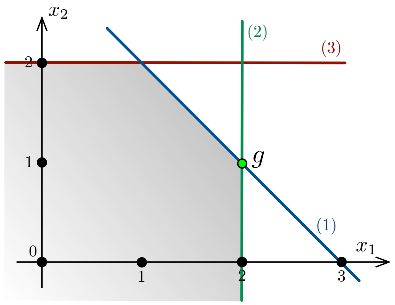

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Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

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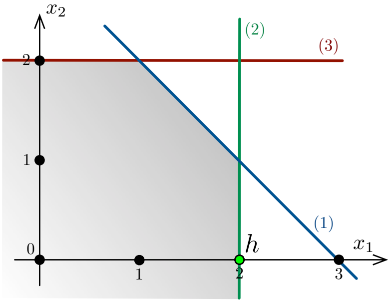

$$P = \left\{ x : \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{A} x \leq \underbrace{\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}}_{b} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}$$

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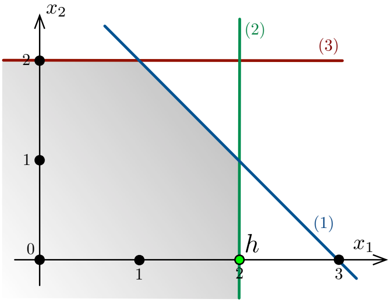

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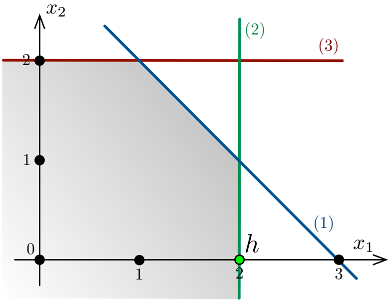

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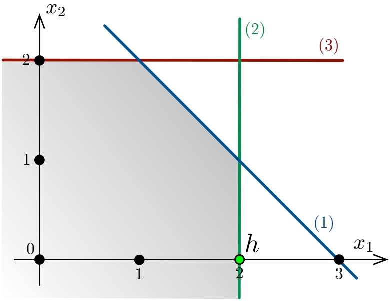

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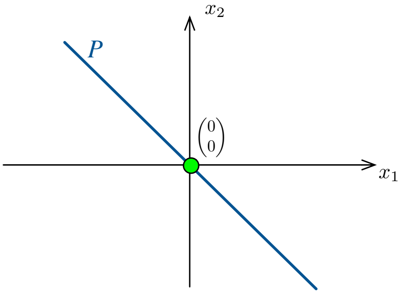

$$P = \left\{ x : \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{A} x \leq \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{b} \right\}$$

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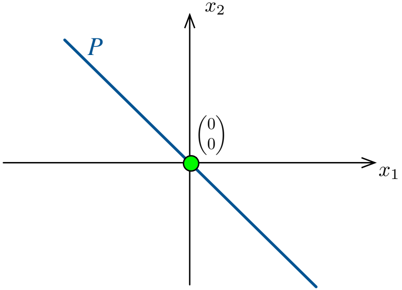

$$P = \left\{ x : \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{A} x \leq \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{b} \right\}$$

$\bar{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has $n = 2$ rows, but $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is NOT extreme.

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

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Let's prove part (1).

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Thus, equality holds throughout $\Rightarrow$ $b = a$ and $c = a$.

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Exercise

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Prove the previous remark.

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Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

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Previous remark implies that $\bar{b} = \bar{A}x^{(1)} = \bar{A}x^{(2)}$.

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Theorem

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Since $rank(\bar{A}) < n$, there exists a non-zero vector d such that $\bar{A}d = \mathbf{0}$.

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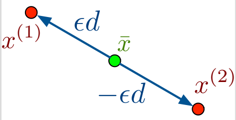

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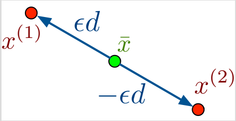

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It suffices to prove the following:

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

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Since $rank(\bar{A}) < n$, there exists a non-zero vector d such that $\bar{A}d = \mathbf{0}$.

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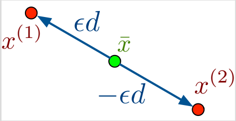

It suffices to prove the following:

(a) $\bar{x}$ is properly contained in the line segment between $x^{(1)}$ and $x^{(2)}$.

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

2. $\boxed{\text{If } rank(\bar{A}) < n\text{, then } \bar{x} \text{ is NOT an extreme point.}}$

Proof

Since $rank(\bar{A}) < n$, there exists a non-zero vector d such that $\bar{A}d = \mathbf{0}$.

Pick a small $\epsilon > 0$.

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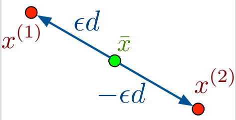

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(a) $\bar{x}$ is properly contained in the line segment between $x^{(1)}$ and $x^{(2)}$.

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Proof

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Why?

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$$\frac{1}{2}x^{(1)} + \frac{1}{2}x^{(2)}$$

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$$\frac{1}{2}x^{(1)} + \frac{1}{2}x^{(2)} = \frac{1}{2}(\bar{x} + \epsilon d) + \frac{1}{2}(\bar{x} - \epsilon d)$$

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$$\frac{1}{2}x^{(1)} + \frac{1}{2}x^{(2)} = \frac{1}{2}(\bar{x} + \epsilon d) + \frac{1}{2}(\bar{x} - \epsilon d) = \bar{x}.$$

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for a small enough ϵ.

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$$P = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$$

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Question

Is $(2, 4, 0)^\top$ an extreme point?

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Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

1. If $rank(\bar{A}) = n$, then $\bar{x}$ is an extreme point.
2. If $rank(\bar{A}) < n$, then $\bar{x}$ is NOT an extreme point.

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$P = \{x : Ax \leq b\}$, where

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$$\bar{A} = \left(\begin{array}{rrr} \color{blue}{1} & \color{blue}{0} & \color{blue}{-1} \\ \color{blue}{0} & \color{blue}{1} & \color{blue}{3} \\ \hline -1 & 0 & 1 \\ 0 & -1 & -3 \\ \hline \color{blue}{0} & \color{blue}{0} & \color{blue}{-1} \end{array}\right)$$

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Since $rank(\bar{A}) = 3$, we know that $(2,4,0)^\top$ is an extreme point!

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Since $rank(\bar{A}) = 3$, we know that $(2, 4, 0)^\top$ is an extreme point!

This is no accident...

Theorem

Let $P = \{x \geq \mathbf{0} : Ax = b\}$ where rows of A are independent. The following are equivalent:

1. $\bar{x}$ is an extreme point of P.
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$\Rightarrow$

The Simplex algorithm moves from extreme points to extreme points.

Simplex - a Geometric Illustration

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$$\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}$$

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Solve using Simplex:

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SOLVE USING SIMPLEX:

- Basis $B = \{3, 4, 5\}$, basic solution $(0, 0, 10, 6, 4)^\top$

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- Basis $B = \{3, 4, 5\}$, basic solution $(0, 0, 10, 6, 4)^\top$
- Basis $B = \{1, 4, 5\}$, basic solution $(5, 0, 0, 1, 9)^\top$

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Solve using Simplex:

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- Basis $B = \{1,4,5\}$, basic solution $(5,0,0,1,9)^\top$
- Basis $B = \{1,2,5\}$, basic solution $(4,2,0,0,6)^\top$

Simplex - a Geometric Illustration

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}}$$

$$P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

SOLVE USING SIMPLEX:

- Basis $B = \{3,4,5\}$, basic solution $(0,0,10,6,4)^\top$
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Simplex - a Geometric Illustration

$$
\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}}
\qquad
P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}
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$$
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Simplex - a Geometric Illustration

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$$\begin{pmatrix} 0 \\ 0 \\ 10 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

However, we cannot draw a picture of this...

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is obtained by adding slack variables to

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Remark

$(0,0,10,6,4)^\top$ extreme point of P_1 $\Rightarrow$ $(0,0)^\top$ extreme point of P_2,

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Remark

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Simplex visits extreme points of P_2 in order:

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$$\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

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$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

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s.t.	
	$x \in P_1$

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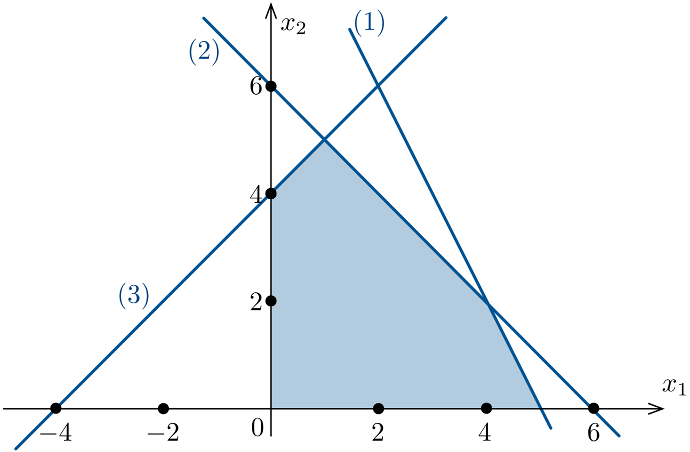

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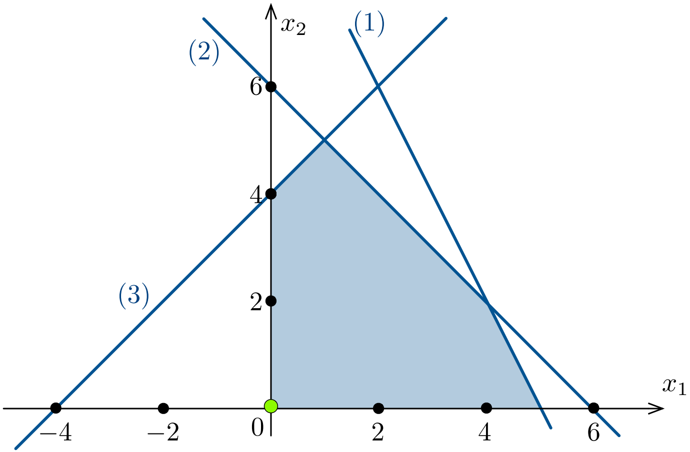

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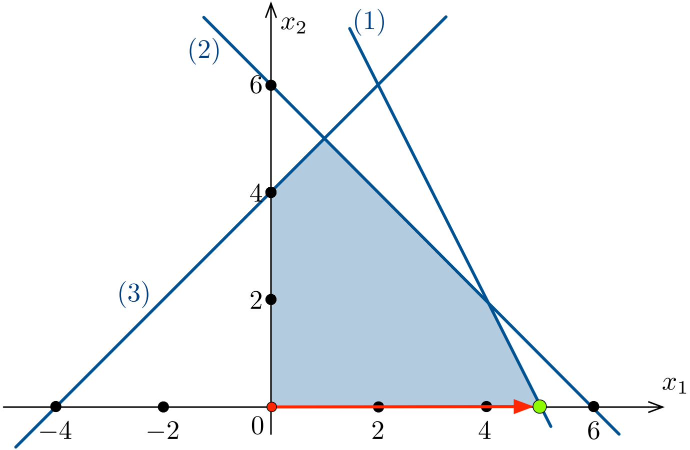

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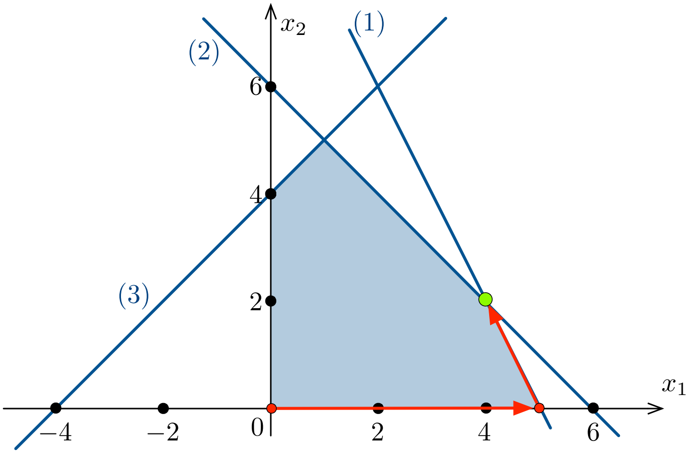

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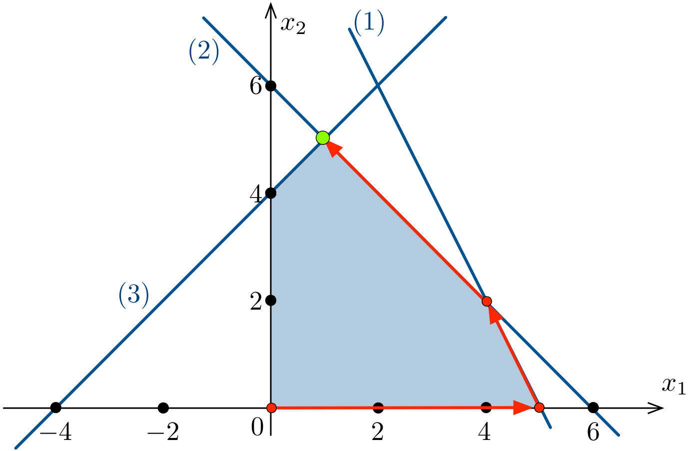

Recap

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- We defined extreme points of convex sets.
- We characterized extreme points in polyhedra.
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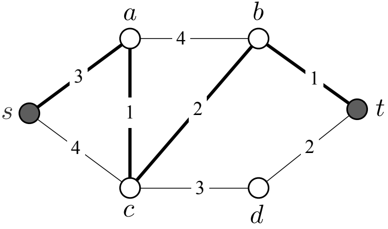
- We showed that the Simplex algorithm moves from extreme point to extreme point.

Module 3: Duality through examples

Recap: Shortest Paths

In an instance of the shortest path problem, we are given

- a graph $G = (V, E)$, a non-negative length c_e for each edge $e \in E$, and
- a pair of vertices s and t in V.



Recap: Shortest Paths

In an instance of the shortest path problem, we are given

- a graph $G = (V, E)$, a non-negative length c_e for each edge $e \in E$, and
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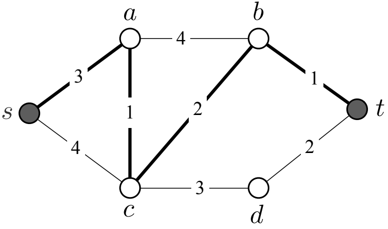
Our goal is to compute an s, t-path P of smallest total length.



Recap: Shortest Paths

In an instance of the shortest path problem, we are given

- a graph $G = (V, E)$, a non-negative length c_e for each edge $e \in E$, and
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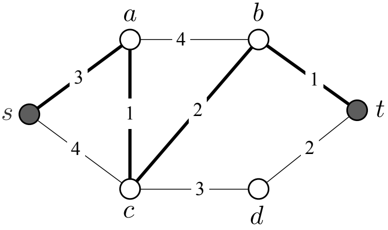
Recall: an s, t-path is a sequence

$$P := u_1u_2, u_2u_3, \ldots, u_{k-1}u_k$$

where

- $u_iu_{i+1} \in E$ for all i, and
- $u_1 = s$, $u_k = t$, and $u_i \neq u_j$ for all $i \neq j$.



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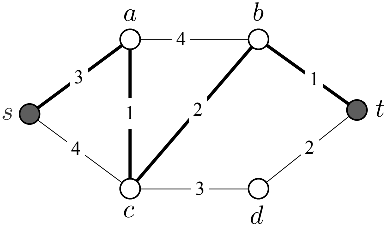


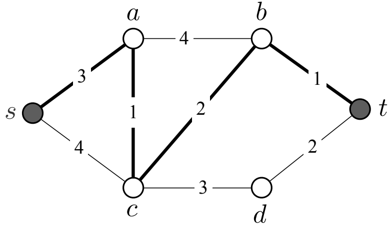
Its length is given by

$$c(P) = c_{u_1u_2} + c_{u_2u_3} + \ldots + c_{u_{k-1}u_k}$$

In the example, we see by inspection that

$$P = sa, ac, cb, bt$$

is a shortest path and that its length is 9.



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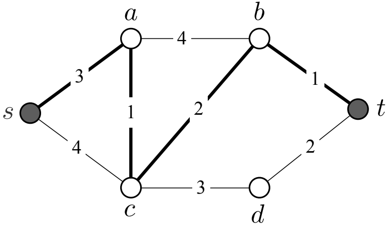

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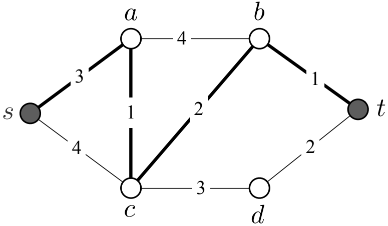

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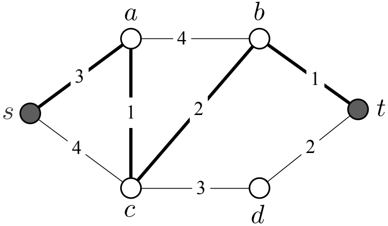

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1. Given a shortest-path instance and a candidate shortest s, t-path P, is there a short proof of its optimality?

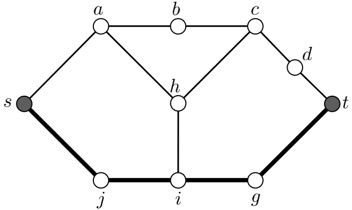
2. How can we find a shortest s, t-path?

We will answer both questions in this module. This lecture focus on question 1.

Shortest Paths: Finding an Intuitive Lower Bound

Cardinality Case

To make our lives easier, we will first consider the cardinality special case of the shortest path problem.



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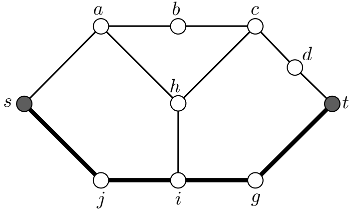

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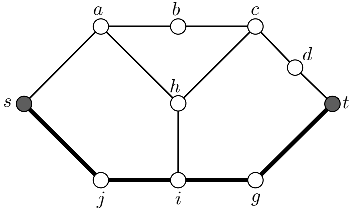

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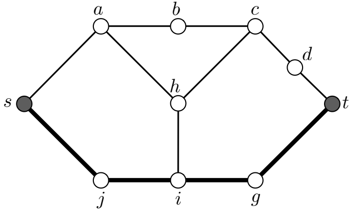

Example: In the diagram above, one easily sees that

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How can we prove this fact?

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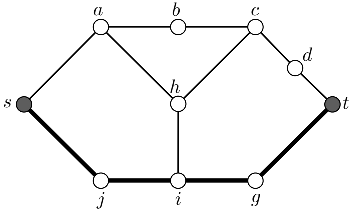

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How can we prove this fact?
$\longrightarrow$ The answer lies in s,t-cuts!

s, t-cuts

Definition

For $U \subseteq V$, we define

$$\delta(U) = \{uv \in E \,:\, u \in U, v \notin U\}$$

and call it an s, t-cut if $s \in U$, and $t \notin U$.

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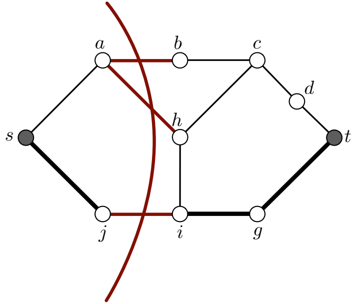

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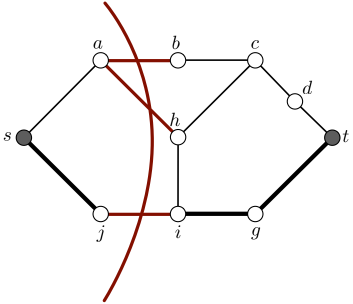

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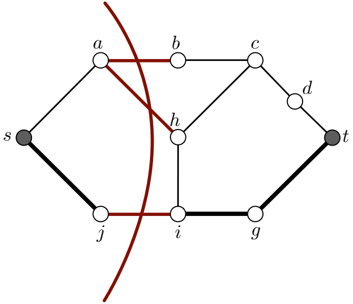
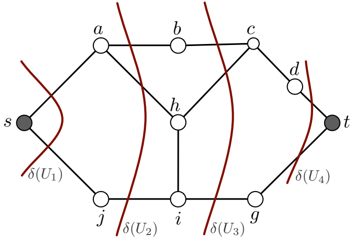


Recall:

- If P is an s,t-path and $\delta(U)$ an s,t-cut, then P contains an edge of $\delta(U)$.
- If $S \subseteq E$ contains an edge from every s,t-cut, then S contains an s,t-path.

From Cuts to Lower-Bounds

The example on the right shows 4 s,t-cuts, $\delta(U_1), \delta(U_2), \delta(U_3), \delta(U_4)$.



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\begin{aligned}
\delta(U_1) &= \{sa, sj\} \\
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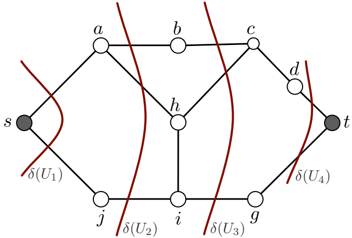

Two important notes:

(1) $\delta(U_i) \cap \delta(U_j) = \emptyset$ for $i \neq j$ and

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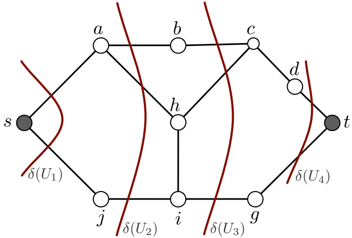

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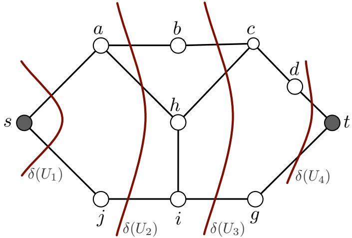

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$\longrightarrow$ Every s,t-path must have at least 4 edges.

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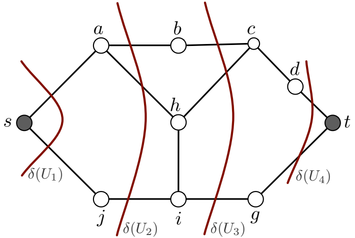

$\longrightarrow$ Every s,t-path must have at least 4 edges.
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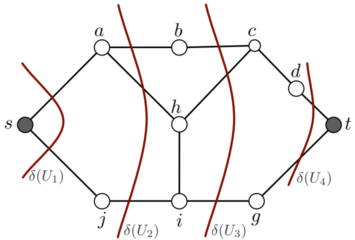
From Cuts to Lower-Bounds

Question

Notice: hi is not in any of the $\delta(U_i)$. Does this mean that hi is not on any shortest s,t-path?



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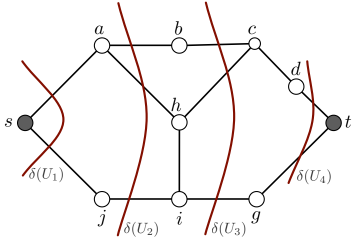

Yes!

An s,t-path that contains hi must also contain an edge from each of the s,t-cuts $\delta(U_i)$.

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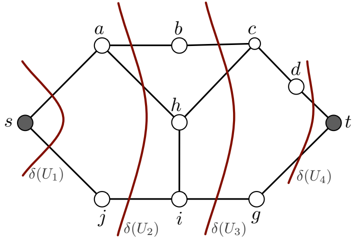

Yes!

An s,t-path that contains hi must also contain an edge from each of the s,t-cuts $\delta(U_i)$. $\longrightarrow$ It must contain at least 5 edges!

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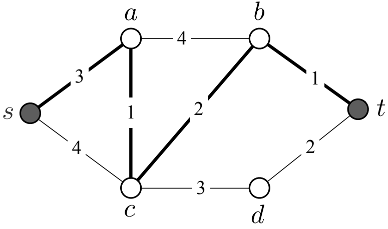
Back to the General Case

In general instances, we assign a non-negative width y_U to every s,t-cut $\delta(U)$.



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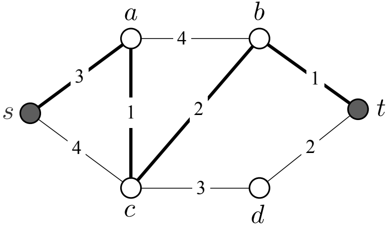

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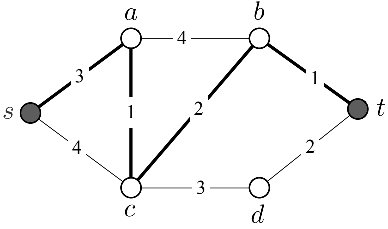

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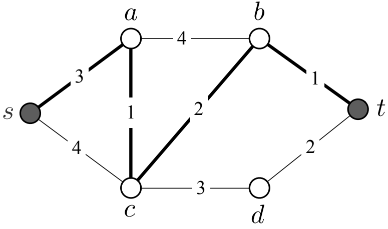

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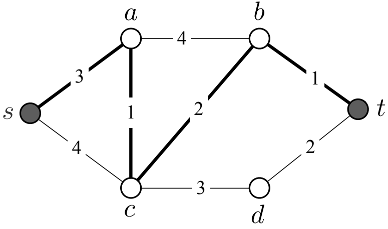

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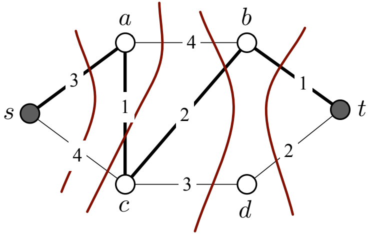
Using math: y is feasible if for all e

$$\sum (y_U : \delta(U) \; s,t\text{-cut and } e \in E) \leq c_e$$

Back to the General Case

Consider the example on the right with 4 s,t-cuts.



$$\begin{aligned} U_1 &= \{s\} \\ U_2 &= \{s,a\} \\ U_3 &= \{s,a,c\} \\ U_4 &= \{s,a,b,c,d\} \end{aligned}$$

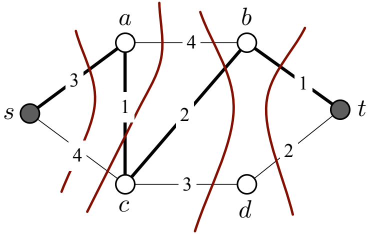
Back to the General Case

Consider the example on the right with 4 s,t-cuts.

The width assignment

$$\begin{aligned} y_{U_1} &= 3 \\ y_{U_2} &= 1 \\ y_{U_3} &= 2 \\ y_{U_4} &= 1 \end{aligned}$$

is easily checked to be feasible.



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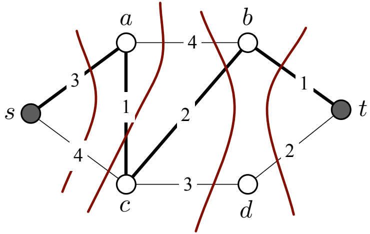
Proposition

If y is a feasible width assignment, then any s,t-path must have length at least

$$\sum(y_U \,:\, U\ s,t\text{-cut}).$$



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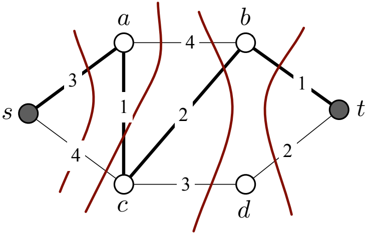

Example:

$$y_{U_1} + y_{U_2} + y_{U_3} + y_{U_4} = 7$$

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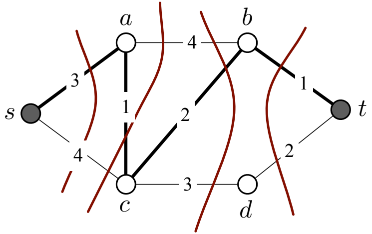

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$\longrightarrow$ Path sa, ac, cb, bt is a shortest path!

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Proof: Consider an s,t-path P. It follows that

$$c(P) \quad = \quad \sum(c_e \ : \ e \in P)$$

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Proof: Consider an s,t-path P. It follows that

$$\begin{aligned} c(P) &= \sum(c_e \,:\, e \in P) \\ &\geq \sum\Big(\sum(y_U \,:\, e \in \delta(U)) \,:\, e \in P\Big) \end{aligned}$$

where the first inequality follows from the feasibility of y.

$\square$

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$$\begin{aligned} c(P) &= \sum (c_e \,:\, e \in P) \\ &\geq \sum \Big(\sum (y_U \,:\, e \in \delta(U)) \,:\, e \in P \Big) \end{aligned}$$

where the first inequality follows from the feasibility of y.

Note: if $\delta(U)$ is an s,t-cut, then P contains at least one edge from $\delta(U)$.

$\square$

Back to the General Case

Proposition

If y is a feasible width assignment, then any s,t-path must have length at least

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Example:

$$y_{U_1} + y_{U_2} + y_{U_3} + y_{U_4} = 7$$

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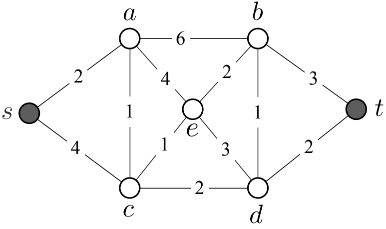
One More Example

Question: Can you spot a shortest s,t-path?



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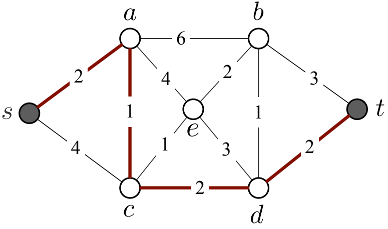

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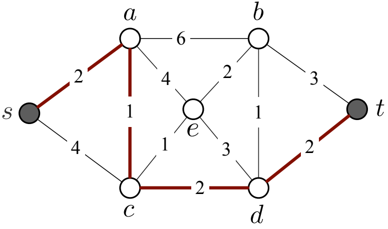

One More Example

Question: Can you spot a shortest s,t-path?
$\longrightarrow$ $P = sa, ac, cd, dt$ of length 7.

Question: Can you prove your guess?
$\longrightarrow$ Yes! There is a feasible dual width assignment of value 7:

$$\begin{aligned}
y_{\{s\}} &= 2\\
y_{\{s,a\}} &= 1\\
y_{\{s,a,c\}} &= 1\\
y_{\{s,a,c,e\}} &= 1\\
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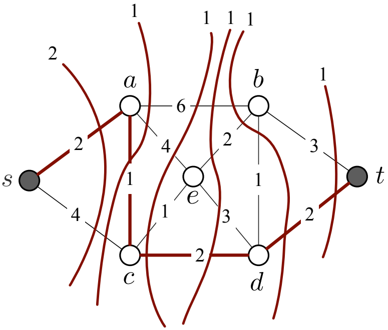
\end{aligned}$$



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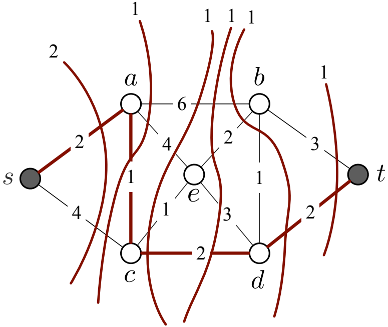
(A) In an instance with a shortest path, can we always find feasible widths to prove optimality?



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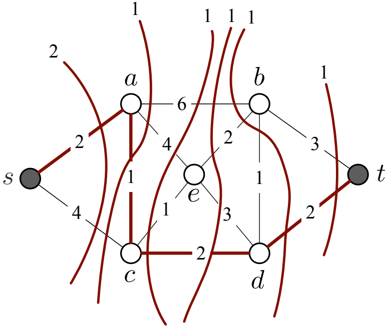
(A) In an instance with a shortest path, can we always find feasible widths to prove optimality?

(B) If so, how do we find a path and these widths?



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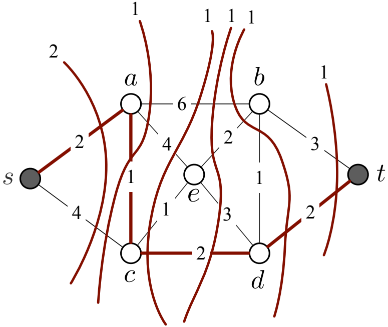

We will answer (A) affirmatively, and provide an efficient algorithm for (B) shortly.

Recap

- A shortest path instance is given by a graph $G = (V, E)$ and non-negative lengths c_e for all $e \in E$.

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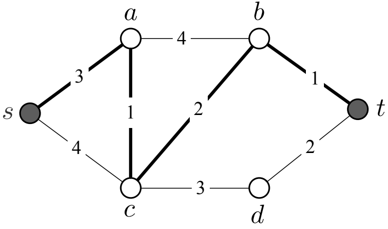
$$c(P) \geq \sum y_U$$

Module 3: Duality through examples (Weak Duality)

Recap: Feasible Widths

Suppose we are given an instance of the shortest path problem. . .



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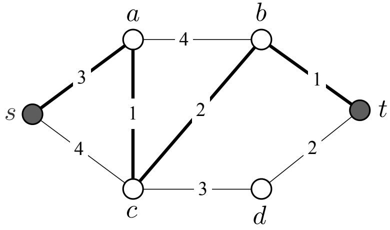

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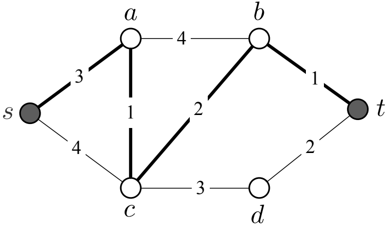

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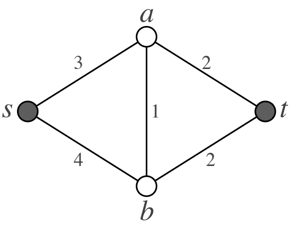

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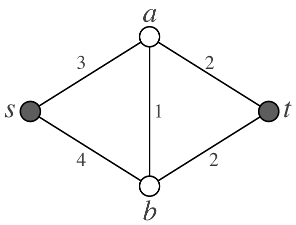

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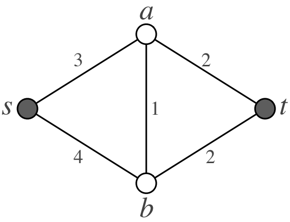

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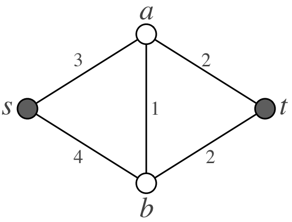

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but, as we will now see, there is a constructive and quite mechanical way to derive the Proposition via linear programming!

An Instructive Example LP

The LP on the right is feasible. . .

$$\begin{array}{ll} \min & (2,3)x \\ \text{s.t.} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

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E.g., $x^1 = (8, 16)^\top$ and $x^2 = (5, 13)^\top$ are feasible.

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Feasible widths provide a lower-bound on the length of a shortest s, t-path...

Question

Can we find a good lower-bound on the objective value of the above LP?

Deriving Valid Inequalities

Let's suppose that x is feasible for the LP on the right.

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Additionally, it satisfies

$$\begin{array}{l} \quad y_1 \cdot (2,1)x \geq y_1 \cdot 20 \\ +\ y_2 \cdot (1,1)x \geq y_2 \cdot 18 \\ +\ y_3 \cdot (-1,1)x \geq y_3 \cdot 8 \\ \hline =\ (2y_1 + y_2 - y_3, y_1 + y_2 + y_3)x \\ \qquad\qquad \geq 20y_1 + 18y_2 + 8y_3 \end{array}$$

for $y_1, y_2, y_3 \geq 0$.

So, if x is feasible for the LP on the right, it also satisfies

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix}$$

for any $y_1, y_2, y_3 \geq 0$.

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Therefore,

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Therefore,

$$\begin{aligned} z(x) &= (2, 3)x \\ &\geq (2, 3)x + 44 - (1, 3)x \\ &= 44 + (1, 0)x \end{aligned}$$

Since $x \geq \mathbb{0}$, it follows that

$$z(x) \geq 44$$

for every feasible solution x!

State of Affairs

We now know that

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Can we find a better lowerbound on $z(x)$ for a feasible x?

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Lowerbounding $z(x)$ Systematically!

We know that a feasible x satisfies

$$0 \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} -$$

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x$$

for any $y_1, y_2, y_3 \geq 0$.

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We know that a feasible x satisfies

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for any $y_1, y_2, y_3 \geq 0$. Therefore,

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} + \left((2, 3) - (y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \right) x \quad (\star)$$

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We want the second term to be non-negative. Since $x \geq \mathbb{0}$, this amounts to choosing y such that

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \leq (2,3)$$

Lowerbounding $z(x)$ Systematically!

We know that a feasible x satisfies

$$0 \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} - (y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x$$

for any $y_1, y_2, y_3 \geq 0$. Therefore,

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} + \left((2,3) - (y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \right) x \quad (\star)$$

We want the second term to be non-negative. Since $x \geq \mathbb{0}$, this amounts to choosing y such that

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \leq (2,3)$$

With such a y we then have from $(\star)$:

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix}$$

Lowerbounding $z(x)$ Systematically!

So, we choose $y \geq \mathbb{0}$ such that

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \leq (2, 3) \quad (\star)$$

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Find the best possible lower-bound on z.

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Find the best possible lower-bound on z. I.e., find $y \geq \mathbb{0}$ such that $(\star)$ holds, and the right-hand side of $(\Diamond)$ is maximized!

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This is a Linear Program:

$$\begin{array}{ll} \max & (20, 18, 8)y \\ \text{s.t.} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} y \leq (2,3) \\ & y \geq \mathbb{0} \end{array}$$

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Solving it gives:

$$\begin{aligned} \bar{y}_1 &= 0 \\ \bar{y}_2 &= 5/2 \\ \bar{y}_3 &= 1/2 \end{aligned}$$

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There is no feasible solution x to

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Since $x^2 = (5, 13)^\top$ is a feasible solution with value 49, it must be optimal!

A General Argument

Suppose now we are given the LP

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq \mathbb{0} \end{aligned}$$

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The best lower-bound on $z(x)$ can be found by the following LP:

$$\begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y \leq c \\ & y \geq \mathbb{0} \end{array}$$

The Dual LP

The linear program is called the dual of primal LP

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

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Theorem

[Weak Duality] If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), then $b^T \bar{y} \leq c^T \bar{x}$.

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$$b^T \bar{y} = \bar{y}^T b \le \bar{y}^T (A\bar{x}) = (A^T \bar{y})^T \bar{x}$$

as $\bar{y} \ge \mathbb{0}$ and $b \le A\bar{x}$,

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as $\bar{y} \ge \mathbb{0}$ and $b \le A\bar{x}$, as $\bar{x} \ge \mathbb{0}$ and $A^T\bar{y} \le c$. $\square$

Lowerbounding the Length of s, t-Paths

Recap: Shortest Path LP

Given a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, the shortest-path LP is

$$
\begin{aligned}
\min \quad & \sum \big(c_e x_e : e \in E\big) \\
\text{s.t.} \quad & \sum \big(x_e \ : \ e \in \delta(U)\big) \geq 1 && (U \subseteq V, s \in U, t \notin U) \\
& x \geq \mathbb{0}, x \text{ integer}
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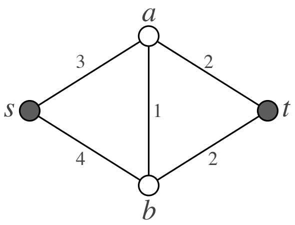
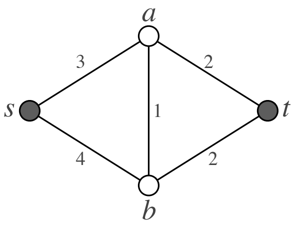
Let's look at an example!

Shortest Path: Example

On the right, we see a sample instance of the shortest-path problem.



Shortest Path: Example



On the right, we see a sample instance of the shortest-path problem.

Here is the corresponding IP:

$$
\begin{array}{lll}
\min & (3,4,1,2,2)x & \\
\text{s.t.} & \begin{array}{l} \\ \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} x \geq \mathbb{1} & \\
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\end{array}
$$

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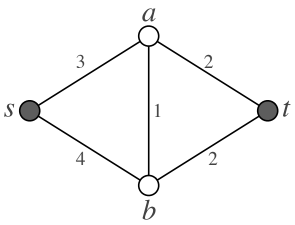

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$x \geq \mathbb{0}, x$ integer

Note that if P is an s,t-path, then letting

$$\bar{x}_e = \begin{cases} 1 & \text{if } e \text{ is an edge of } P \\ 0 & \text{otherwise.} \end{cases}$$

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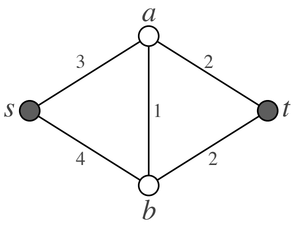

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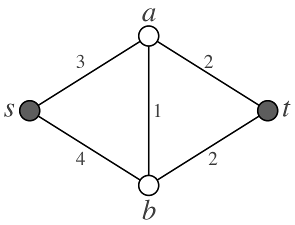

Example:

$$P = sa, ab, bt$$

is an s,t-path.

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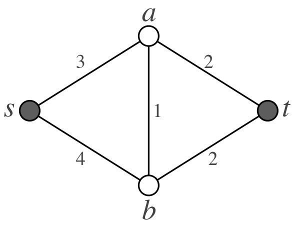

Example:

$$P = sa, ab, bt$$

is an s,t-path.

$$x = (1,0,1,0,1)^T$$

is feasible for the IP, and its value is 6.

$$\min \quad (3,4,1,2,2)x$$

$$\text{s.t.} \quad \begin{array}{c|c} & \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{array}{c} \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} x \geq \mathbb{1} \end{array}$$

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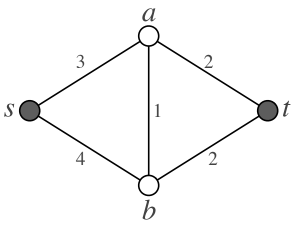


Example:

$$P = sa, ab, bt$$

is an s,t-path.

$$x = (1,0,1,0,1)^T$$

is feasible for the IP, and its value is 6.

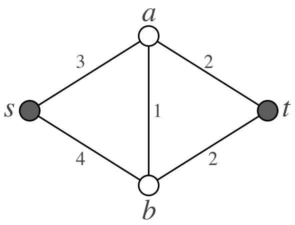
Remark

The optimal value of the shortest path IP is, at most, the length of a shortest s,t-path.

$$\min \quad (3,4,1,2,2)x \qquad \text{(P)}$$

$$\text{s.t.} \quad \begin{array}{c} \\ \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} x \geq \mathbb{1}$$

$$x \geq \mathbb{0}, x \; \cancel{\text{integer}}$$



Note that dropping the integrality restriction can not increase the optimal value.

$$\min \quad (3,4,1,2,2)x \qquad \text{(P)}$$

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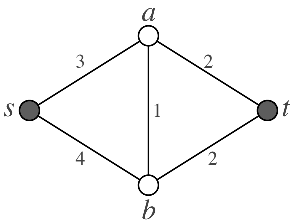


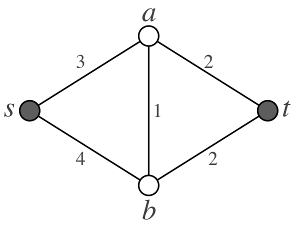
Note that dropping the integrality restriction can not increase the optimal value.

The resulting LP is called the linear programming relaxation of the IP.

$$\min \quad (3,4,1,2,2)x \qquad \text{(P)}$$

$$\text{s.t.} \quad \begin{array}{c|ccccc} & sa & sb & ab & at & bt \\ \hline \{s\} & 1 & 1 & 0 & 0 & 0 \\ \{s,a\} & 0 & 1 & 1 & 1 & 0 \\ \{s,b\} & 1 & 0 & 1 & 0 & 1 \\ \{s,a,b\} & 0 & 0 & 0 & 1 & 1 \end{array} \; x \geq \mathbb{1}$$

$x \geq \mathbb{0}, x$ ~~integer~~



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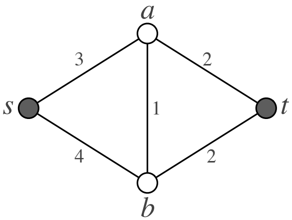
Straight from Weak Duality theorem, we have that:

Remark

The dual of (P) has optimal value no larger than that of (P)!

The dual of the shortest path LP on the previous slide is given by



$$\max \quad \mathbb{1}^\top y$$

$$\text{s.t.} \quad \begin{array}{c} \\ sa \\ sb \\ ab \\ at \\ bt \end{array} \begin{array}{c} \begin{array}{cccc} \{s\} & \{s,a\} & \{s,b\} & \{s,a,b\} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} y \leq \begin{pmatrix} 3 \\ 4 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$y \geq \mathbb{0}$$

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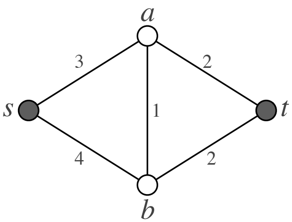

$$
\begin{array}{ll}
\max & \mathbb{1}^\top y \\
\text{s.t.} & \begin{array}{c} \\ sa \\ sb \\ ab \\ at \\ bt \end{array}
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& y \ge \mathbb{0}
\end{array}
$$

Note that dual solutions assign the value $y_U \ge 0$ to every s,t-cut $\delta(U)$!

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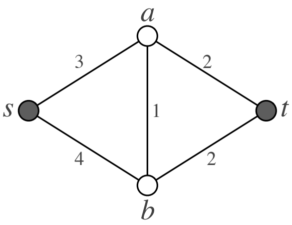

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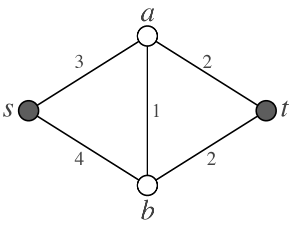

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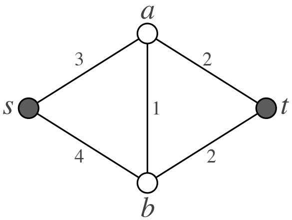

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Remark

y is feasible for the above LP if and only if it is a feasible width assignment for the s,t-cuts in the given shortest path instance!

General Shortest Path Instances

Input: $G = (V, E)$, $c_e \geq 0$ for all $e \in E$, $s, t \in V$.

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Feasible solutions to (D) correspond precisely to feasible width assignments. Weak duality implies that $\sum y_U$ is, at most, the length of a shortest s,t-path!

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- The dual LP of

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Module 3: Duality through examples (Shortest Path Algorithm)

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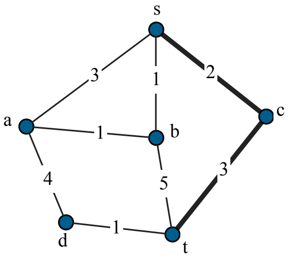
The figure on the right shows another simple instance of the shortest s, t-path problem.



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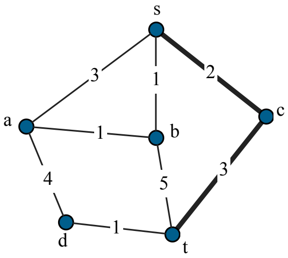

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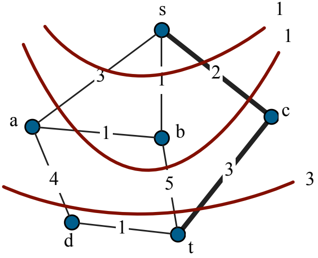

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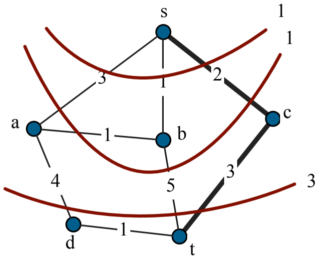

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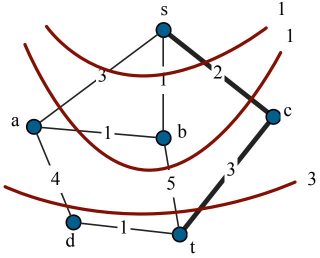

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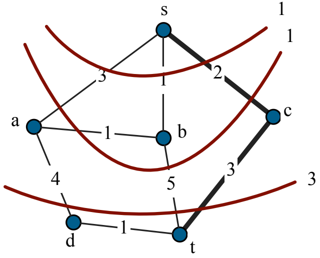

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Letting

$$x_e = \begin{cases} 1 & e \textbf{ bold} \text{ in figure} \\ 0 & \text{otherwise} \end{cases}$$

for all $e \in E$ is feasible for shortest path LP.

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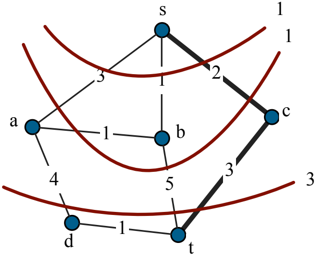

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$$y_{\{s\}} = y_{\{s,b\}} = 1, \; y_{\{s,a,b,c\}} = 3,$$

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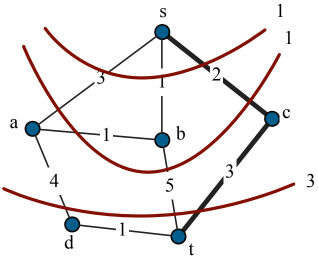

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If $\bar{x}$ is feasible for shortest path LP, and $\bar{y}$ is feasible for its dual then $b^T\bar{y} \leq c^T\bar{x}$.

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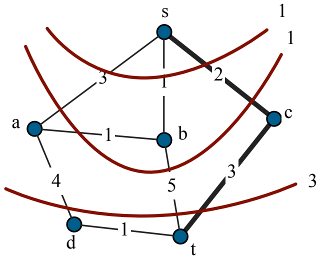

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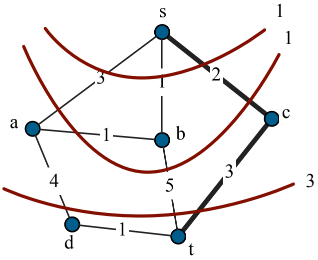

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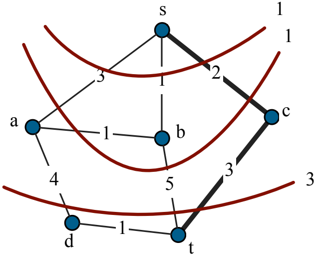

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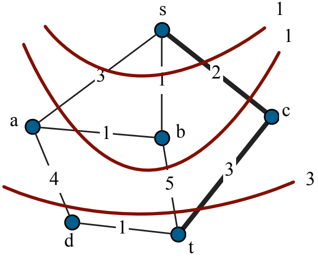

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1. How did we find the bold path?
2. How did we find the dual solution?
3. Is there always a shortest s, t-path and a dual solution whose value matches its length?

An Algorithm for the Shortest s, t-Path Problem

Arcs and Directed Paths

So far: edges of a graph $G = (V, E)$ are unordered pairs of vertices.

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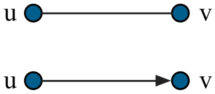

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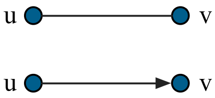
A directed path is then a sequence of arcs:

$$\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \ldots, \overrightarrow{v_{k-1} v_k},$$

where $\overrightarrow{v_i v_{i+1}}$ is an arc in the given graph, and $v_i \neq v_j$ for all $i \neq j$.



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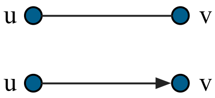

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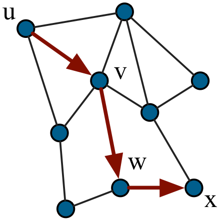


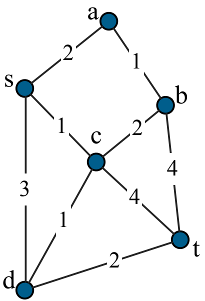
where $\overrightarrow{v_iv_{i+1}}$ is an arc in the given graph, and $v_i \neq v_j$ for all $i \neq j$.

Example:

$$\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{wx}$$

is a directed u, x-path.

Shortest Paths: Algorithmic Ideas



Idea: Find an s,t-path P and a feasible dual y s.t. $c(P) = \mathbb{1}^T y$. How?

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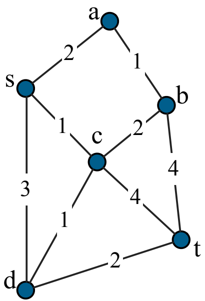

Definition

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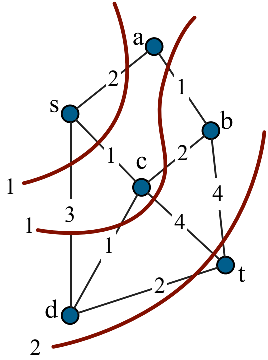

Examples: for the dual y given on the right,

- $\text{slack}_y(sa) =$

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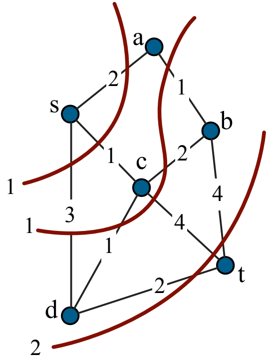

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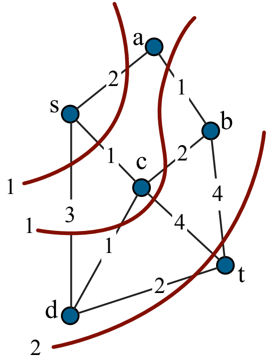

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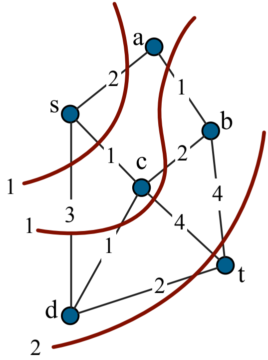

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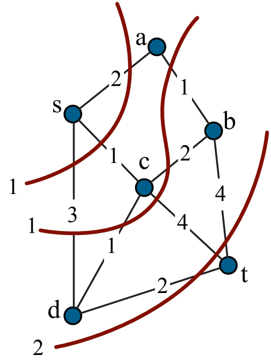

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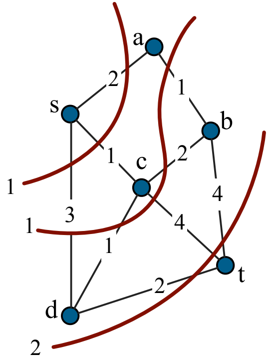

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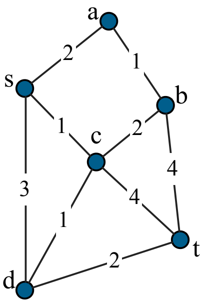

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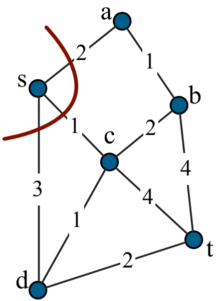

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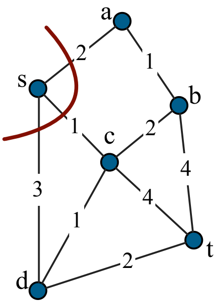

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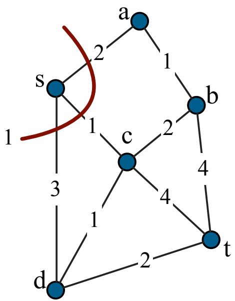

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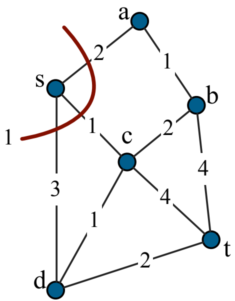

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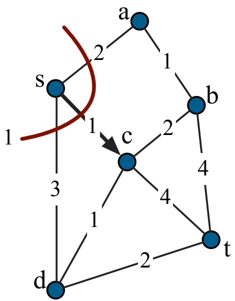

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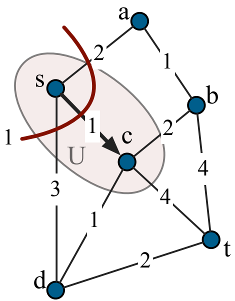

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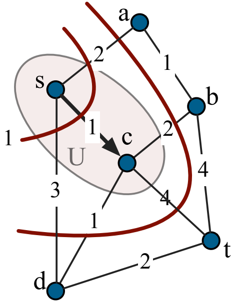

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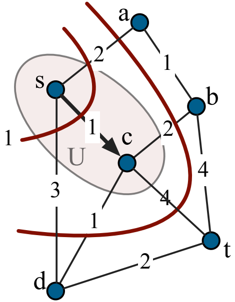

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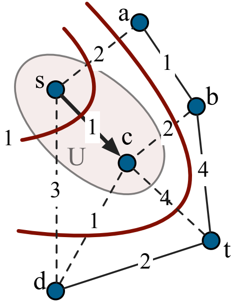

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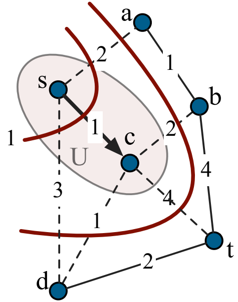

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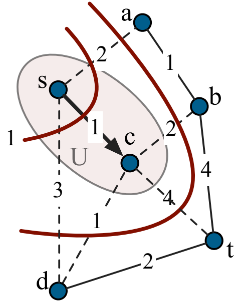

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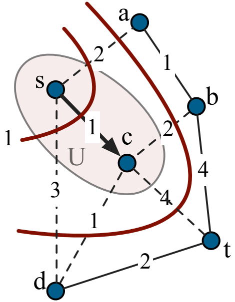

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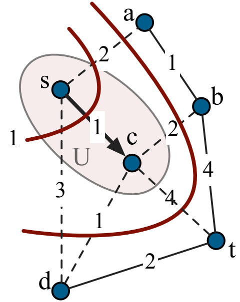

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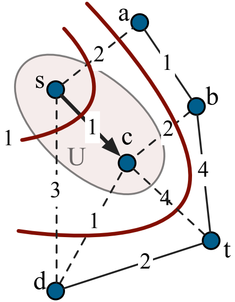

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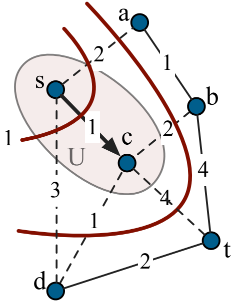

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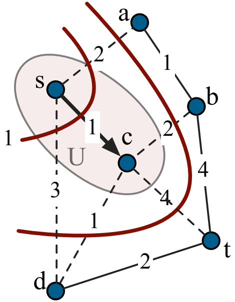
Edges cd and sa minimize slack. Pick one arbitrarily: sa.



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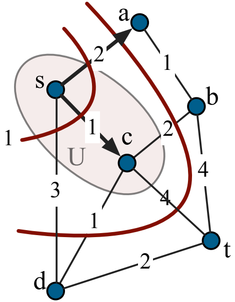
Set $y_U = \text{slack}_y(sa) = 1$ and convert sa into arc $\overrightarrow{sa}$



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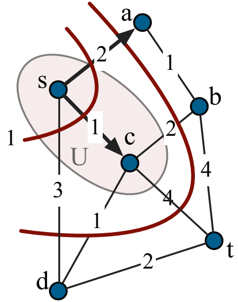
Shortest Paths: Building Duals Incrementally

Q: Which vertices are reachable from s via directed paths?



$$\max \quad \sum(y_S \,:\, \delta(S)\ s,t\text{-cut})$$

$$\text{s.t.} \quad \sum(y_S \,:\, e \in \delta(S)) \leq c_e \quad (e \in E)$$

$$y \geq \mathbb{0}$$

Shortest Paths: Building Duals Incrementally

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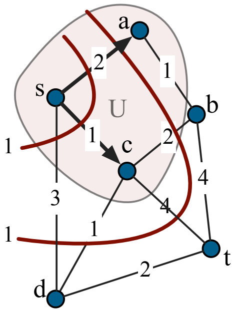

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Shortest Paths: Building Duals Incrementally

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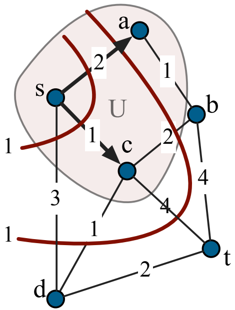
Natural idea: Increase $y_{\{s,a,c\}}$ by as much as we can. How much?



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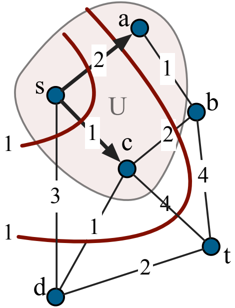

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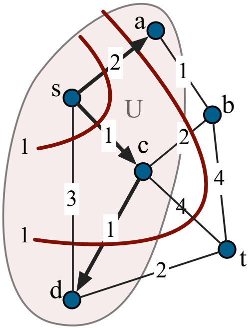
Also: change cd into $\overrightarrow{cd}$, and let

$$U = \{s, a, c, d\}$$

be the reachable vertices from s



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Vertices reachable from s by directed paths:

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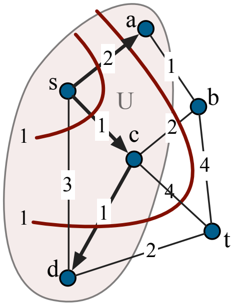

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\mathsf{slack}_y(ab) &= \\
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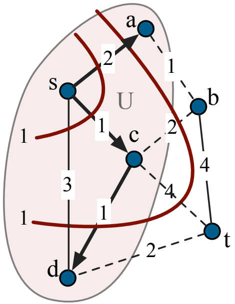

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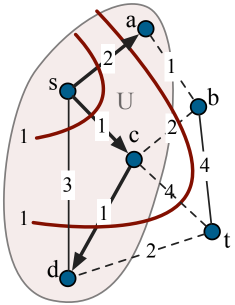
Shortest Paths: Building Duals Incrementally

Vertices reachable from s by directed paths:

$$U = \{s, a, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned} \mathsf{slack}_y(ab) &= 1 \\ \mathsf{slack}_y(cb) &= \\ \mathsf{slack}_y(ct) &= \\ \mathsf{slack}_y(dt) &= \end{aligned}$$



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\text{slack}_y(ab) &= 1 \\
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\text{slack}_y(ct) &= \\
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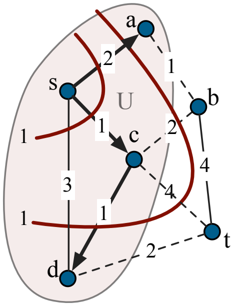

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$$\begin{aligned}
\text{slack}_y(ab) &= 1 \\
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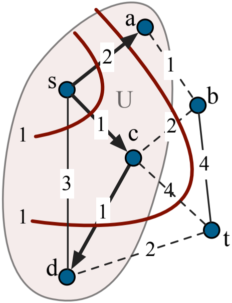

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Shortest Paths: Building Duals Incrementally

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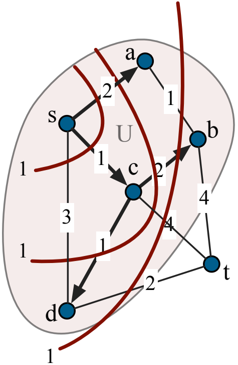
\mathsf{slack}_y(ct) &= 4 - 1 = 3 \\
\mathsf{slack}_y(dt) &= 2
\end{aligned}$$

Let $y_{\{s,a,c,d\}} = 1$, add equality arc $\overrightarrow{cb}$, and update the set

$$U = \{s, a, b, c, d\}$$

of vertices reachable from s



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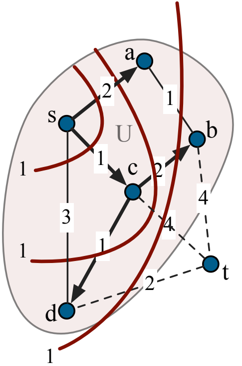

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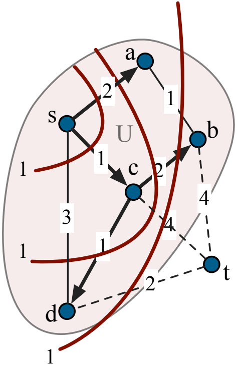

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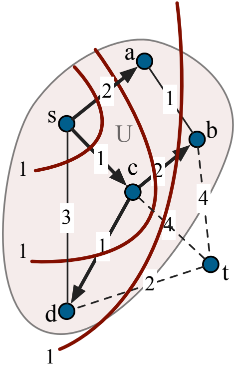
$$\begin{aligned} \mathsf{slack}_y(bt) &= 4 \\ \mathsf{slack}_y(ct) &= \\ \mathsf{slack}_y(dt) &= \end{aligned}$$



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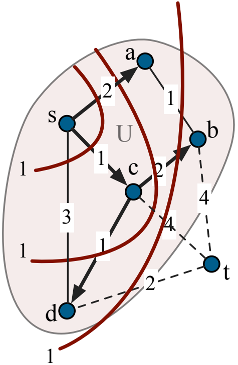

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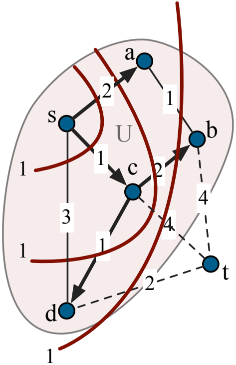

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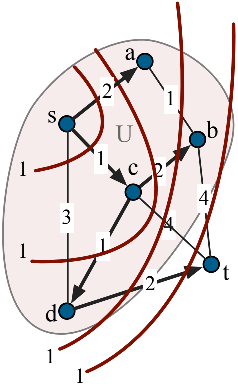

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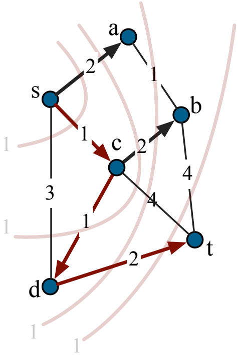
Shortest Paths: Building Duals Incrementally

Note: we now have a directed s, t-path in our graph:

$$P = \overrightarrow{sc}, \overrightarrow{cd}, \overrightarrow{dt},$$

and its length is 4!



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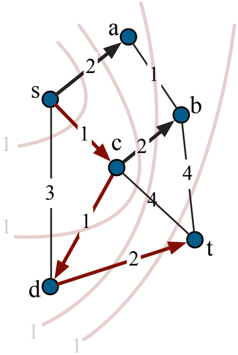
and $y_U = 0$ otherwise.



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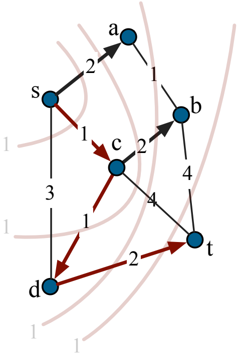

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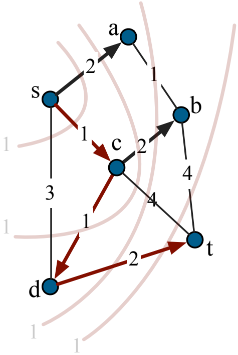
$\longrightarrow$ Path P is a shortest path!



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Shortest Path Algorithm

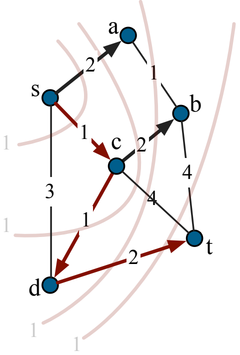
To compute the shortest Path for the instance on the right, we used the following algorithm:

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

$y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$
while $t \notin U$ **do**
    Let ab be an edge in $\delta(U)$ of smallest slack for y where $a \in U$, $b \notin U$
    $y_U := \text{slack}_y(ab)$
    $U := U \cup \{b\}$
    change edge ab into an arc $\overrightarrow{ab}$
end while
return A directed st-path P.



Recap

- We saw a shortest path algorithm that computes
  (a) an s, t-path P, and
  (b) a feasible solution y for the dual of the shortest path LP

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- Have a look at the book. It has another full example run of the shortest path algorithm

Module 3: Duality through examples (Correctness Shortest Path Algorithm)

Recap: Shortest Path Algorithm

Previous lecture: we showed an algorithm for the shortest path problem that computes

- An s,t-path P

Shortest path LP:

$$\begin{aligned} \min \quad & \sum (c_e x_e \,:\, e \in E) \\ \text{s.t.} \quad & \sum (x_e \,:\, e \in \delta(S)) \geq 1 \\ & \qquad\qquad (\delta(S) \ s,t\text{-cut}) \\ & x \geq \mathbb{0} \end{aligned}$$

Shortest path dual:

$$\begin{aligned} \max \quad & \sum (y_S \,:\, \delta(S) \ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \leq c_e \\ & \qquad\qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

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Previous lecture: we showed an algorithm for the shortest path problem that computes

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Important: $c^T x = \mathbb{1}^T y$

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Important: $c^T x = \mathbb{1}^T y \rightarrow P$ is a shortest path!

Shortest path LP:

$$\begin{array}{ll} \min & \sum(c_e x_e \,:\, e \in E) \\ \text{s.t.} & \sum(x_e \,:\, e \in \delta(S)) \geq 1 \\ & \qquad (\delta(S)\ s,t\text{-cut}) \\ & x \geq \mathbb{0} \end{array}$$

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Previous lecture: we showed an algorithm for the shortest path problem that computes

- An s,t-path P whose characteristic vector, x^P, is feasible for the shortest path LP, and
- a feasible solution, y, for the dual of the shortest path LP.

Important: $c^T x = \mathbb{1}^T y \rightarrow P$ is a shortest path!

We will start this lecture with another example!

Shortest path LP:

$$
\begin{aligned}
\min \quad & \sum (c_e x_e \,:\, e \in E) \\
\text{s.t.} \quad & \sum (x_e \,:\, e \in \delta(S)) \geq 1 \\
& \qquad\qquad (\delta(S)\ s,t\text{-cut}) \\
& x \geq \mathbb{0}
\end{aligned}
$$

Shortest path dual:

$$
\begin{aligned}
\max \quad & \sum (y_S \,:\, \delta(S)\ s,t\text{-cut}) \\
\text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \leq c_e \\
& \qquad\qquad (e \in E) \\
& y \geq \mathbb{0}
\end{aligned}
$$

Recall the algorithm we developed previously:

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

```
y_W := 0 for all st-cuts δ(W). Set U := {s}
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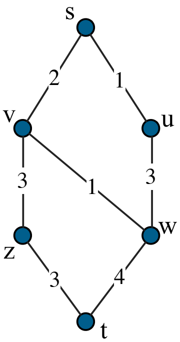

$\longrightarrow$ Run this on the example instance on the right.

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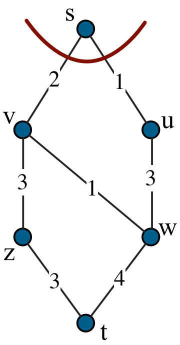

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Initially: $y = \mathbb{0}$ and $U = \{s\}$

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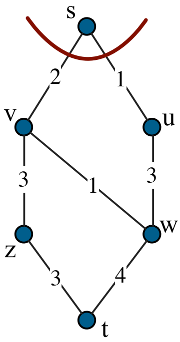
Step 1 su edge with smallest slack in $\delta(U)$



Initially: $y = \mathbb{0}$ and $U = \{s\}$

Step 1 $\quad su$ edge with smallest slack in $\delta(U)$
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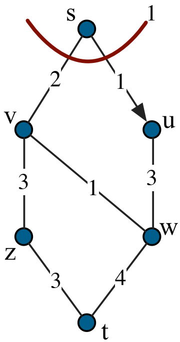

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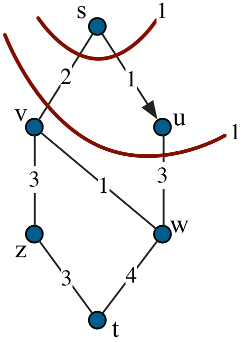
Slack-minimal edge is sv



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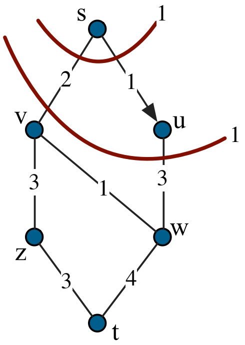

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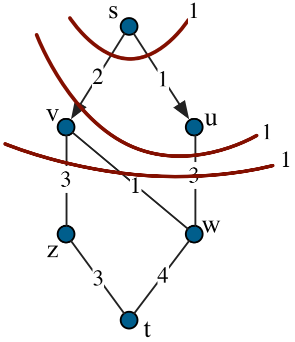

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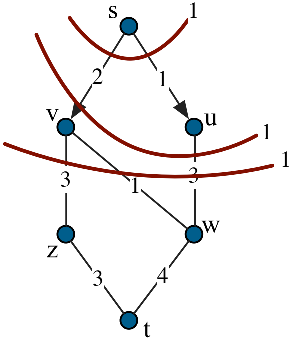

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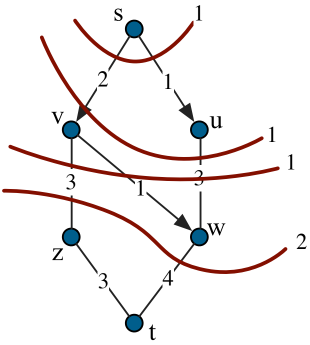

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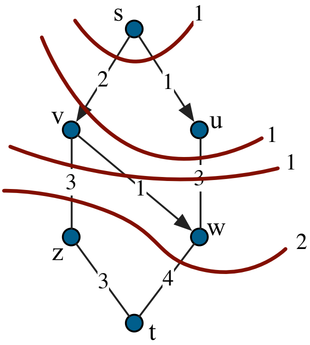

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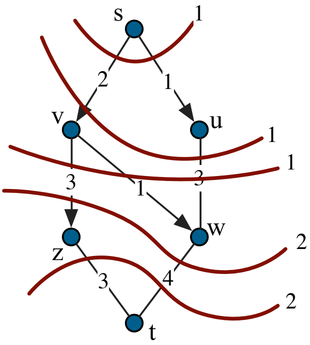

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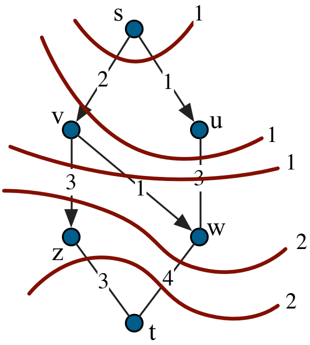

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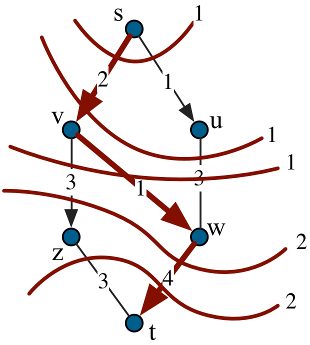

Now: We have a directed s,t-path P of length 7,

Initially: $y = \mathbb{0}$ and $U = \{s\}$

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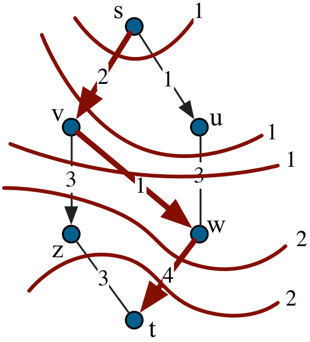

Now: We have a directed s,t-path P of length 7, and a dual feasible solution of the same value!

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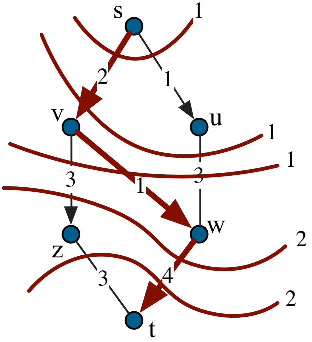
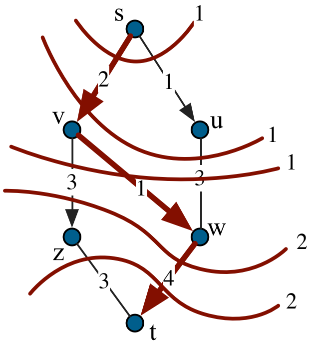


Now: We have a directed s, t-path P of length 7, and a dual feasible solution of the same value!

$\longrightarrow$ P is a shortest path!

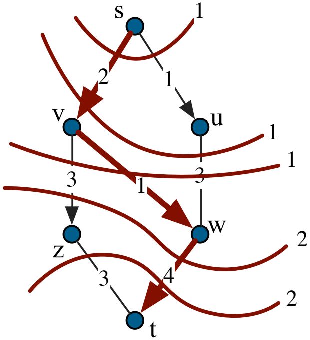
Question

Will the algorithm always terminate?



Question

Will the algorithm always terminate? Will it always find an s, t-path P whose length is equal to the value of a feasible dual solution?



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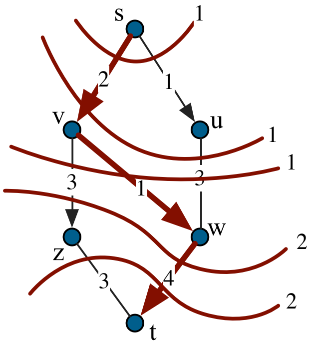


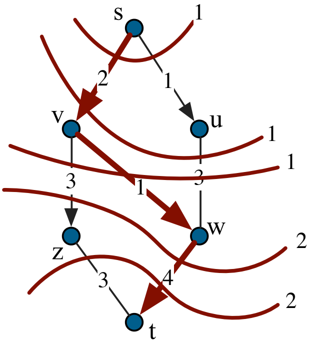
This lecture: We will show the answers to the above are yes & yes!

Revisited: Shortest Path Optimality Conditions

Recall: the slack of an edge $uv \in E$ for a feasible dual solution y is

$$c_{uv} - \sum (y_U \,:\, e \in \delta(U))$$

.



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Recall: the slack of an edge $uv \in E$ for a feasible dual solution y is

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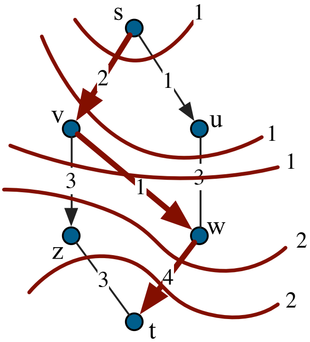
We call an edge $uv \in E$ an equality edge if its slack is 0.



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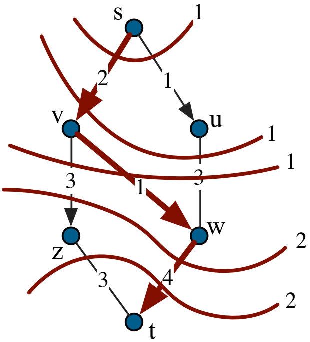
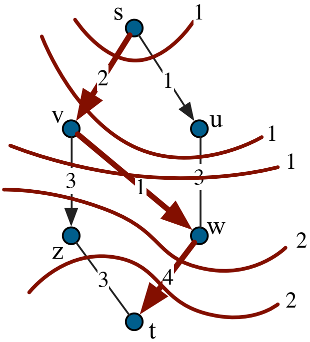
Example: edge vz is an equality edge, and zt is not!



We will also call a cut $\delta(U)$ active for a dual solution y if $y_U > 0$.



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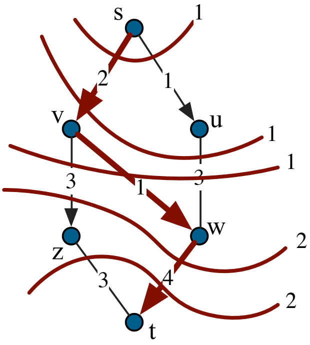
Example: $\delta(\{s, v, u\})$ is active, while $\delta(\{s, v\})$ is not!



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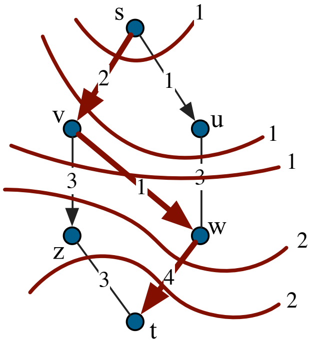
Proposition

Let y be a feasible dual solution, and P and s,t-path. P is a shortest path if



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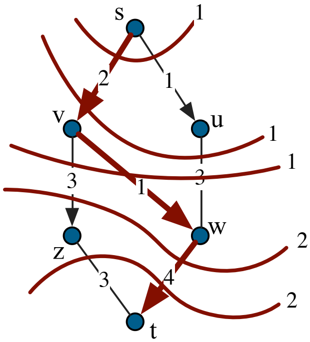
Let y be a feasible dual solution, and P and s,t-path. P is a shortest path if

(i) all edges on P are equality edges, and



Revisited: Shortest Path Optimality Conditions

Proposition

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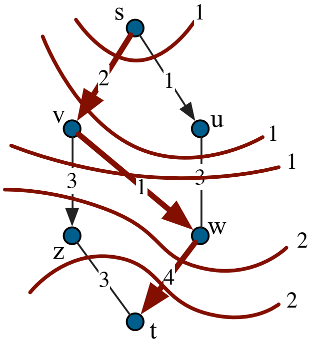

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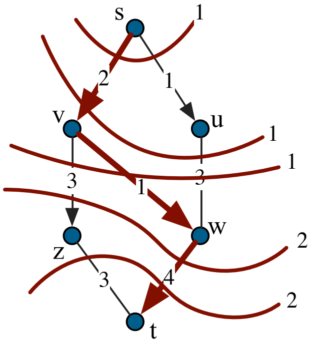

Note: Both conditions are satisfied in the example on the right.

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$$\sum_{e \in P} c_e = \sum_{e \in P} (\sum(y_U \,:\, e \in \delta(U))$$

because every edge on P is an equality edge by (i).

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But, by (ii), $y_U > 0$ only if $|P \cap \delta(U)| = 1$. Hence:

$$\sum_{e \in P} c_e = \sum_U y_U$$

$\square$

Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

1: $y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$
2: **while** $t \notin U$ **do**
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Note: The algorithm terminates since one vertex is added to U in every step and V is finite.

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The Shortest Path Algorithm maintains throughout its execution that:

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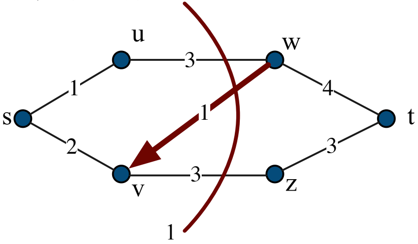

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(I3) no active cut $\delta(U)$ has an entering arc: an arc wu with $w \notin U$, and $u \in U$,

Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.
Output: A shortest st-path P

$y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$
while $t \notin U$ **do**
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  change edge ab into an arc $\overrightarrow{ab}$
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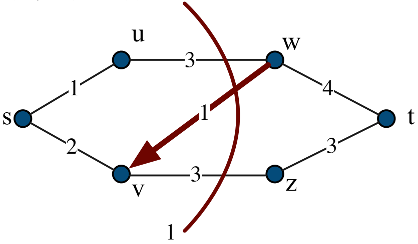
return A directed st-path P.

Note: The algorithm terminates since one vertex is added to U in every step and V is finite.



It suffices to show:

Proposition

The Shortest Path Algorithm maintains throughout its execution that:

(I1) y is a feasible dual,

(I2) arcs are equality arcs (i.e., have 0 slack),

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Correctness of the Shortest Path Algorithm

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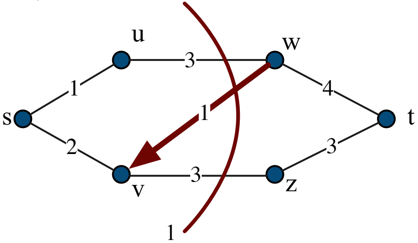

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To show: $\delta(U)$ active $\longrightarrow$ P has exactly one edge in $\delta(U)$.

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Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

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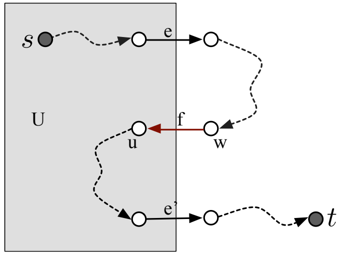
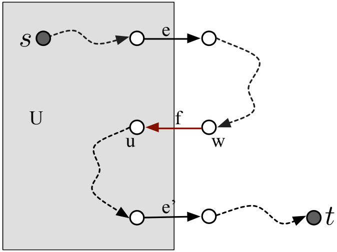


Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

Let e and e' be the first two edges on P that leave $\delta(U)$.



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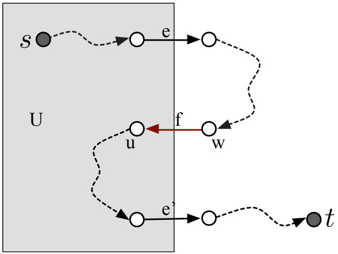
For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

Let e and e' be the first two edges on P that leave $\delta(U)$.

Then, there must also be an arc f on P that enters U,



Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

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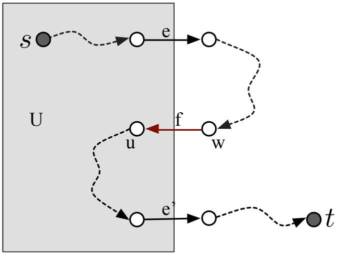
Then, there must also be an arc f on P that enters U, but this contradicts (I3)!



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Correctness of the Shortest Path Algorithm

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Let's now prove the proposition!

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Let's now prove the proposition!

Trivial: (I1) – (I5) hold after Step 1.
Suppose (I1) – (I5) hold before Step 3.

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Let's now prove the proposition!

Trivial: (I1) – (I5) hold after Step 1.
Suppose (I1) – (I5) hold before Step 3.
We will show that they also hold after Step 6.

(I1) y is Dual Feasible

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$$\max \quad \sum (y_S \;:\; \delta(S) \; s,t\text{-cut})$$

$$\text{s.t.} \quad \sum (y_S \;:\; e \in \delta(S)) \leq c_e \quad (e \in E)$$

$$y \geq \mathbb{0}$$

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$$\begin{aligned} \max \quad & \sum(y_S \,:\, \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum(y_S \,:\, e \in \delta(S)) \leq c_e \\ & \qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

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The smallest slack of any of these constraints is precisely the increase in y_U.

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Also: The constraint of the newly created arc holds with equality after the increase
$\longrightarrow$ (I2) continues to hold and constraints for arcs have slack 0.

Correctness of the Shortest Path Algorithm

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- the only new active cut created is $\delta(U)$

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- the only new active cut created is $\delta(U)$
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$\longrightarrow$ (I3) holds after Step 6

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Correctness of the Shortest Path Algorithm

Note: Algorithms adds arc ab in current step, and (I4) implies that there is a directed s, a-path P.

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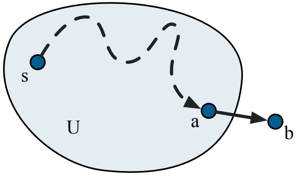

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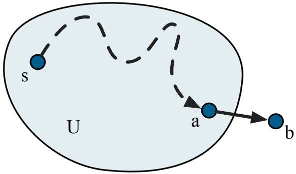

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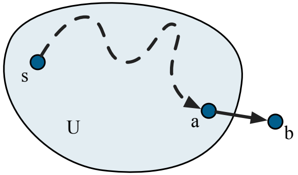

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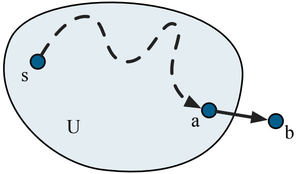

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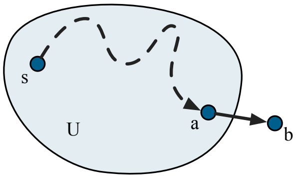

Finally, the only new arc added is ab. As b is added to U, (I5) continues to hold.

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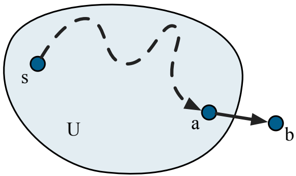

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We are now done!

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Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

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1: y_W := 0 for all st-cuts δ(W). Set U := {s}
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- Moreover, the length of P equals the objective value of y, and hence, P must be a shortest s,t-path.
- Implicitly, we therefore showed that the shortest path LP always has an optimal integer solution!

Module 4: Duality Theory (Weak Duality)

Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible $0, 1$-solutions for the LP

$$\min \sum \big(c_e x_e : e \in E\big)$$

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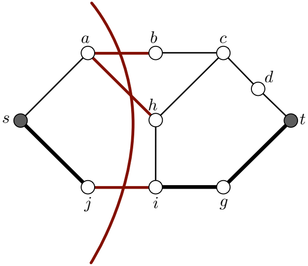
$$(U \subseteq V, s \in U, t \notin U)$$

$$x \geq \mathbb{0}$$



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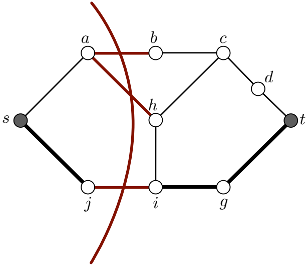

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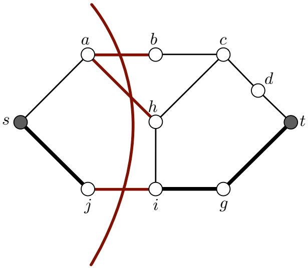

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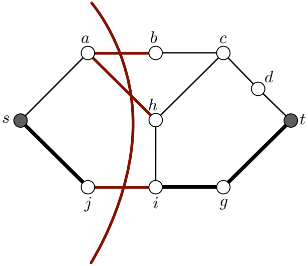

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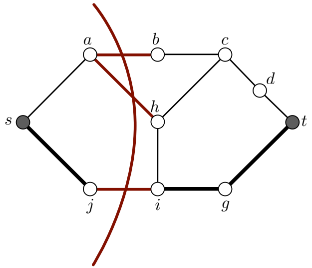

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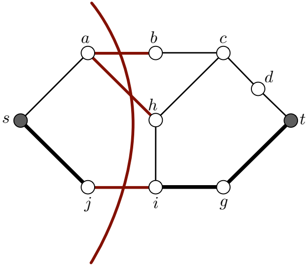

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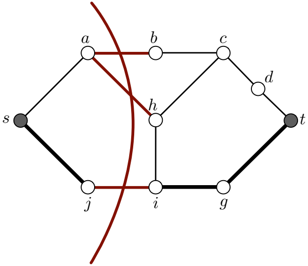

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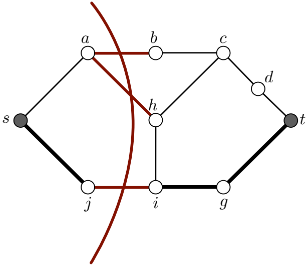

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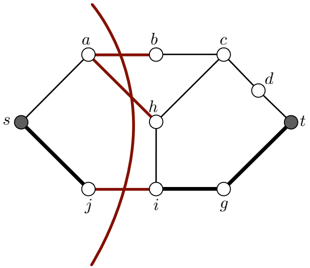

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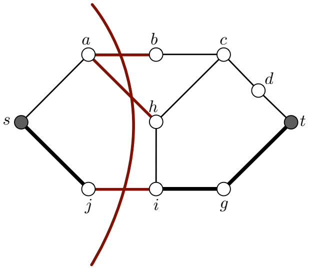

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A: As before:

$$\begin{aligned} \text{primal variables} &\equiv \text{dual constraints} \\ \text{primal constraints} &\equiv \text{dual variables} \end{aligned}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$					$(\mathrm{P}_{\min})$
		$\leq$ constraint	≥ 0 variable		
max	$c^\top x$	$=$ constraint	free variable	min	$b^\top y$
subject to		$\geq$ constraint	≤ 0 variable	subject to	
	$Ax \; ? \; b$	≥ 0 variable	$\geq$ constraint		$A^\top y \; ? \; c$
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Its dual LP:

$$\min \ (3,4)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \; ? \; \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

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The following table shows how constraints and variables in primal and dual LPs correspond:

$(P_{\max})$			$(P_{\min})$
$\max\ c^\top x$ subject to $Ax ? b$, $x ? \mathbb{0}$	$\leq$ constraint	≥ 0 variable	$\min\ b^\top y$ subject to $A^\top y ? c$, $y ? \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 1:

$$\max\ (1,0,2)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \geq 0, x_3 \text{ free}$$

Its dual LP:

$$\min\ (3,4)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y\ ?\ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
max $\quad c^\top x$ subject to $Ax \, ? \, b$ $x \, ? \, \mathbb{0}$	$\leq$ constraint $=$ constraint $\geq$ constraint ≥ 0 variable free variable ≤ 0 variable	≥ 0 variable free variable ≤ 0 variable $\geq$ constraint $=$ constraint $\leq$ constraint	min $\quad b^\top y$ subject to $A^\top y \, ? \, c$ $y \, ? \, \mathbb{0}$

Example 1:

$$\max\ (1,0,2)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$x_1, x_2 \geq 0, x_3$ free

Its dual LP:

$$\min\ (3,4)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \begin{matrix} \geq \\ \geq \\ = \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$y_1 \geq 0, y_2$ free

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

(P_{max})		(P_{min})	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax ? b$	≥ 0 variable	$\geq$ constraint	$A^\top y ? c$
$x ? \mathbb{0}$	free variable	$=$ constraint	$y ? \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\min d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \; ? \; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	free variable	$=$ constraint	$y \; ? \; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\min\ d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

To compute dual LP, check right-hand side of table:

$$\max\ e^T x \qquad \text{(D)}$$
$$\text{s.t. } Wx \; ? \; d$$
$$x \; ? \; \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \, ? \, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \, ? \, c$
$x \, ? \, \mathbb{0}$	free variable	$=$ constraint	$y \, ? \, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\begin{aligned} &\min\ d^T y \\ &\text{s.t. } W^T y \geq e \\ &y \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

To compute dual LP, check right-hand side of table:

$$\begin{aligned} &\max\ e^T x \\ &\text{s.t. } Wx \ ? \ d \\ &x \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
max $c^\top x$ subject to $Ax\ ?\ b$, $x\ ?\ \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y\ ?\ c$, $y\ ?\ \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\min\ d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

To compute dual LP, check right-hand side of table:

$$\max\ e^T x \qquad \text{(D)}$$
$$\text{s.t. } Wx \leq d$$
$$x \geq \mathbb{0}$$

Primal-Dual Pairs

Example 2:

$$\begin{aligned} \min\ & d^T y \\ \text{s.t. } & W^T y \geq e \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\begin{aligned} \max\ & e^T x \\ \text{s.t. } & W x \leq d \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Primal-Dual Pairs

Example 2:

$$\min\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \geq b$$
$$x \geq \mathbb{0}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\max\ b^T x \qquad \text{(D)}$$
$$\text{s.t. } A^T y \leq c$$
$$y \geq \mathbb{0}$$

Primal-Dual Pairs

Example 2:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\begin{aligned} \max\ & b^T x \\ \text{s.t. } & A^T y \leq c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

This is consistent with the earlier discussion we had!

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(P_{\max})$				
max $c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} \mathbb{0}$	
	$=$ constraint	free variable		
	$\geq$ constraint	≤ 0 variable		
	≥ 0 variable	$\geq$ constraint		
	free variable	$=$ constraint		
	≤ 0 variable	$\leq$ constraint		

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$	
max $c^\top x$ subject to $Ax \, ? \, b$ $x \, ? \, \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \, ? \, c$ $y \, ? \, \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \ ? \ \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y \ ? \ \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(P_{\max})$			
max $c^\top x$ subject to $Ax \, ? \, b$, $x \, ? \, \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \, ? \, c$, $y \, ? \, \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Header of the last two columns: $(P_{\min})$

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t. } \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t. } \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \; ? \; \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$	
max $c^\top x$ subject to $Ax \mathbin{?} b$, $x \mathbin{?} \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbin{?} c$, $y \mathbin{?} \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
max $c^\top x$	$\leq$ constraint	≥ 0 variable	min $b^\top y$
subject to	$=$ constraint	free variable	subject to
Ax **?** b	$\geq$ constraint	≤ 0 variable	$A^\top y$ **?** c
x **?** $\mathbb{0}$	≥ 0 variable	$\geq$ constraint	y **?** $\mathbb{0}$
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Theorem

Let $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$ represent the above.

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \; ? \; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	free variable	$=$ constraint	$y \; ? \; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Theorem

Let $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$ represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T \bar{x} \leq b^T \bar{y}$$

If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for $(\mathrm{P}_{\max})$, and $\bar{y}$ is optimal for $(\mathrm{P}_{\min})$.

Primal-Dual Pairs

Theorem

Let (P_{max}) and (P_{min}) represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

Example 3 (continued):

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \text{ free}$$

Feasible solutions: $\bar{x} = (5, -3, 0)^T$ and $\bar{y} = (0, 4, -2)^T$.

Primal-Dual Pairs

Theorem

Let (P_{max}) and (P_{min}) represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

Example 3 (continued):

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \text{ free}$$

Feasible solutions: $\bar{x} = (5, -3, 0)^T$ and $\bar{y} = (0, 4, -2)^T$.
Since $(12, 26, 20)\bar{x} = (-2, 2, 13)\bar{y} = -18 \longrightarrow$ both are optimal!

Proving the General Weak Duality Theorem

(P_{max})					
max $c^\top x$ subject to $Ax\ ?\ b$, $x\ ?\ \mathbb{0}$		$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y\ ?\ c$, $y\ ?\ \mathbb{0}$	
		$=$ constraint	free variable		
		$\geq$ constraint	≤ 0 variable		
		≥ 0 variable	$\geq$ constraint		
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

General Primal LP:

$$\max c^T x$$
$$\text{s.t. } \mathrm{row}_i(A)x \leq b_i \ (i \in R_1)$$
$$\mathrm{row}_i(A)x \geq b_i \ (i \in R_2)$$
$$\mathrm{row}_i(A)x = b_i \ (i \in R_3)$$
$$x_j \geq 0 \ (j \in C_1)$$
$$x_j \leq 0 \ (j \in C_2)$$
$$x_j \text{ free } (j \in C_3)$$

Proving the General Weak Duality Theorem

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$		
max	$c^\top x$	$\leq$ constraint	≥ 0 variable	min	$b^\top y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \mathbin{?} b$	$\geq$ constraint	≤ 0 variable		$A^\top y \mathbin{?} c$
	$x \mathbin{?} \mathbb{0}$	≥ 0 variable	$\geq$ constraint		$y \mathbin{?} \mathbb{0}$
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

General Primal LP:

$$\begin{aligned} &\max c^T x \\ &\text{s.t. } \mathrm{row}_i(A)x \leq b_i \ (i \in R_1) \\ &\quad \mathrm{row}_i(A)x \geq b_i \ (i \in R_2) \\ &\quad \mathrm{row}_i(A)x = b_i \ (i \in R_3) \\ &\quad x_j \geq 0 \ (j \in C_1) \\ &\quad x_j \leq 0 \ (j \in C_2) \\ &\quad x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} &\min b^T y \\ &\text{s.t. } \mathrm{col}_j(A)^T y \geq c_j \ (j \in C_1) \\ &\quad \mathrm{col}_j(A)^T y \leq c_j \ (j \in C_2) \\ &\quad \mathrm{col}_j(A)^T y = c_j \ (j \in C_3) \\ &\quad y_i \geq 0 \ (i \in R_1) \\ &\quad y_i \leq 0 \ (i \in R_2) \\ &\quad y_i \text{ free } (i \in R_3) \end{aligned}$$

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } \mathsf{row}_i(A)x \le b_i\ \ (i \in R_1) \\ &\qquad \mathsf{row}_i(A)x \ge b_i\ \ (i \in R_2) \\ &\qquad \mathsf{row}_i(A)x = b_i\ \ (i \in R_3) \\ &\qquad x_j \ge 0\ (j \in C_1) \\ &\qquad x_j \le 0\ (j \in C_2) \\ &\qquad x_j\ \text{free}\ (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} &\min\ b^T y \\ &\text{s.t. } \mathsf{col}_j(A)^T y \ge c_j\ \ (j \in C_1) \\ &\qquad \mathsf{col}_j(A)^T y \le c_j\ \ (j \in C_2) \\ &\qquad \mathsf{col}_j(A)^T y = c_j\ \ (j \in C_3) \\ &\qquad y_i \ge 0\ (i \in R_1) \\ &\qquad y_i \le 0\ (i \in R_2) \\ &\qquad y_i\ \text{free}\ (i \in R_3) \end{aligned}$$

We can rewrite the above LPs using slack variables!

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \qquad (\star) \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned}$$

Proving the General Weak Duality Theorem

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & Ax + s = b \\
& s_i \geq 0 \ (i \in R_1) \\
& s_i \leq 0 \ (i \in R_2) \\
& s_i = 0 \ (i \in R_3) \\
& x_j \geq 0 \ (j \in C_1) \\
& x_j \leq 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & A^T y + w = c \qquad (\star) \\
& w_j \leq 0 \ (j \in C_1) \\
& w_j \geq 0 \ (j \in C_2) \\
& w_j = 0 \ (j \in C_3) \\
& y_i \geq 0 \ (i \in R_1) \\
& y_i \leq 0 \ (i \in R_2) \\
& y_i \text{ free } (i \in R_3)
\end{aligned}
$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \qquad (\star) \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned}$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$.

Proving the General Weak Duality Theorem

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & Ax + s = b \\
& s_i \geq 0 \ (i \in R_1) \\
& s_i \leq 0 \ (i \in R_2) \\
& s_i = 0 \ (i \in R_3) \\
& x_j \geq 0 \ (j \in C_1) \\
& x_j \leq 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & A^T y + w = c \qquad (\star) \\
& w_j \leq 0 \ (j \in C_1) \\
& w_j \geq 0 \ (j \in C_2) \\
& w_j = 0 \ (j \in C_3) \\
& y_i \geq 0 \ (i \in R_1) \\
& y_i \leq 0 \ (i \in R_2) \\
& y_i \text{ free } (i \in R_3)
\end{aligned}
$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s}$$

Proving the General Weak Duality Theorem

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & Ax + s = b \\
& s_i \geq 0 \ (i \in R_1) \\
& s_i \leq 0 \ (i \in R_2) \\
& s_i = 0 \ (i \in R_3) \\
& x_j \geq 0 \ (j \in C_1) \\
& x_j \leq 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & A^T y + w = c \qquad (\star) \\
& w_j \leq 0 \ (j \in C_1) \\
& w_j \geq 0 \ (j \in C_2) \\
& w_j = 0 \ (j \in C_3) \\
& y_i \geq 0 \ (i \in R_1) \\
& y_i \leq 0 \ (i \in R_2) \\
& y_i \text{ free } (i \in R_3)
\end{aligned}
$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$
\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} \overset{(\star)}{=} (c - \bar{w})^T\bar{x} + \bar{y}^T\bar{s} = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}.
$$

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \qquad (\star) \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned}$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} \overset{(\star)}{=}$$
$$(c - \bar{w})^T\bar{x} + \bar{y}^T\bar{s} = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}.$$

We can show that $\bar{w}^T\bar{x} \leq 0$ and $\bar{y}^T\bar{s} \geq 0$

Proving the General Weak Duality Theorem

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & Ax + s = b \\
& s_i \geq 0 \ (i \in R_1) \\
& s_i \leq 0 \ (i \in R_2) \\
& s_i = 0 \ (i \in R_3) \\
& x_j \geq 0 \ (j \in C_1) \\
& x_j \leq 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & A^T y + w = c \qquad (\star) \\
& w_j \leq 0 \ (j \in C_1) \\
& w_j \geq 0 \ (j \in C_2) \\
& w_j = 0 \ (j \in C_3) \\
& y_i \geq 0 \ (i \in R_1) \\
& y_i \leq 0 \ (i \in R_2) \\
& y_i \text{ free } (i \in R_3)
\end{aligned}
$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$
\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} \overset{(\star)}{=} (c - \bar{w})^T\bar{x} + \bar{y}^T\bar{s} = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}.
$$

We can show that $\bar{w}^T\bar{x} \leq 0$ and $\bar{y}^T\bar{s} \geq 0 \longrightarrow \quad \bar{y}^T b \geq c^T\bar{x}$

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

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If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

Consequences of Weak Duality

Theorem

Let ($\mathsf{P}_{\mathsf{max}}$) and ($\mathsf{P}_{\mathsf{min}}$) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T \bar{x} \leq b^T \bar{y}$$

If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for ($\mathsf{P}_{\mathsf{max}}$), and $\bar{y}$ is optimal for ($\mathsf{P}_{\mathsf{min}}$).

(i) ($\mathsf{P}_{\mathsf{max}}$) is unbounded $\longrightarrow$ ($\mathsf{P}_{\mathsf{min}}$) infeasible

(ii) ($\mathsf{P}_{\mathsf{min}}$) is unbounded $\longrightarrow$ ($\mathsf{P}_{\mathsf{max}}$) infeasible

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T \bar{x} \leq b^T \bar{y}$$

If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}).

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}). By weak duality $\longrightarrow$ $c^T\bar{x} \leq b^T\bar{y}$ for all $\bar{x}$ feasible for (P_{max}), and hence the latter is bounded.

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}). By weak duality $\longrightarrow$ $c^T\bar{x} \leq b^T\bar{y}$ for all $\bar{x}$ feasible for (P_{max}), and hence the latter is bounded.

(ii) Similar to (i)

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T \bar{x} \leq b^T \bar{y}$$

If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}). By weak duality $\longrightarrow$ $c^T \bar{x} \leq b^T \bar{y}$ for all $\bar{x}$ feasible for (P_{max}), and hence the latter is bounded.

(ii) Similar to (i)

(iii) weak duality $\longrightarrow$ both (P_{max}) and (P_{min}) bounded

Consequences of Weak Duality

Theorem

Let ($P_{\max}$) and ($P_{\min}$) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for ($P_{\max}$), and $\bar{y}$ is optimal for ($P_{\min}$).

(i) ($P_{\max}$) is unbounded $\longrightarrow$ ($P_{\min}$) infeasible

(ii) ($P_{\min}$) is unbounded $\longrightarrow$ ($P_{\max}$) infeasible

(iii) ($P_{\max}$) and ($P_{\min}$) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for ($P_{\min}$). By weak duality $\longrightarrow$ $c^T\bar{x} \leq b^T\bar{y}$ for all $\bar{x}$ feasible for ($P_{\max}$), and hence the latter is bounded.

(ii) Similar to (i)

(iii) weak duality $\longrightarrow$ both ($P_{\max}$) and ($P_{\min}$) bounded

Fundamental Theorem of LP $\longrightarrow$ Both LPs must have an optimal solution! $\square$

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \; ? \; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	free variable	$=$ constraint	$y \; ? \; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Recap

- We can use the above table to compute duals of general LPs

(P_{max})		(P_{min})	
max $c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} 0$	$\leq$ constraint $=$ constraint $\geq$ constraint ≥ 0 variable free variable ≤ 0 variable	≥ 0 variable free variable ≤ 0 variable $\geq$ constraint $=$ constraint $\leq$ constraint	min $b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} 0$

Recap

- We can use the above table to compute duals of general LPs
- Weak duality theorem: if $\bar{x}$ and $\bar{y}$ are feasible for (P_{max}) and (P_{min}), then:

$$c^T\bar{x} \leq b^T\bar{y}$$

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
max $c^\top x$ subject to $Ax \;?\; b$ $x \;?\; \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \;?\; c$ $y \;?\; \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Recap

- We can use the above table to compute duals of general LPs
- Weak duality theorem: if $\bar{x}$ and $\bar{y}$ are feasible for $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$, then:

$$c^T \bar{x} \leq b^T \bar{y}$$

Both are optimal if equality holds!

Module 4: Duality Theory (Strong Duality)

Recap: Weak Duality

$(\mathrm{P}_{\max})$					
max	$c^\top x$	$\leq$ constraint	≥ 0 variable	min	$b^\top y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \mathrel{?} b$	$\geq$ constraint	≤ 0 variable		$A^\top y \mathrel{?} c$
	$x \mathrel{?} \mathbb{0}$	≥ 0 variable	$\geq$ constraint		$y \mathrel{?} \mathbb{0}$
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

Header: $(\mathrm{P}_{\max})$ spans the left three columns; $(\mathrm{P}_{\min})$ spans the right three columns.

Last lecture: we described a method to construct the dual of a general linear program.

Recap: Weak Duality

$(\mathrm{P}_{\max})$				
max $c^\top x$ subject to $Ax \,?\, b$, $x \,?\, \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \,?\, c$, $y \,?\, \mathbb{0}$	$(\mathrm{P}_{\min})$
	$=$ constraint	free variable		
	$\geq$ constraint	≤ 0 variable		
	≥ 0 variable	$\geq$ constraint		
	free variable	$=$ constraint		
	≤ 0 variable	$\leq$ constraint		

Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

Recap: Weak Duality

$(\mathrm{P}_{\max})$				
max $c^\top x$ subject to $Ax \mathbin{?} b$, $x \mathbin{?} 0$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbin{?} c$, $y \mathbin{?} 0$	$(\mathrm{P}_{\min})$
	$=$ constraint	free variable		
	$\geq$ constraint	≤ 0 variable		
	≥ 0 variable	$\geq$ constraint		
	free variable	$=$ constraint		
	≤ 0 variable	$\leq$ constraint		

Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right – a max LP that falls in the left $(\mathrm{P}_{\max})$ part of the table.

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

Recap: Weak Duality

($P_{\max}$)			($P_{\min}$)
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \mathbin{?} b$	≥ 0 variable	$\geq$ constraint	$A^\top y \mathbin{?} c$
$x \mathbin{?} \mathbf{0}$	free variable	$=$ constraint	$y \mathbin{?} \mathbf{0}$
	≤ 0 variable	$\leq$ constraint	

Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right – a max LP that falls in the left ($P_{\max}$) part of the table.

$\longrightarrow$ The dual of (P) is a min LP.

$$\max\ (2,-1,3)x \tag{P}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

Recap: Weak Duality

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
max $\quad c^\top x$	$\leq$ constraint	≥ 0 variable	min $\quad b^\top y$
subject to	$=$ constraint	free variable	subject to
$Ax \mathbin{?} b$	$\geq$ constraint	≤ 0 variable	$A^\top y \mathbin{?} c$
$x \mathbin{?} \mathbb{0}$	≥ 0 variable	$\geq$ constraint	$y \mathbin{?} \mathbb{0}$
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \ ? \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$y \ ? \ \mathbb{0}$$

Recap: Weak Duality

(P_{max})			(P_{min})
max $c^\top x$	$\leq$ constraint	≥ 0 variable	min $b^\top y$
subject to	$=$ constraint	free variable	subject to
$Ax \,?\, b$	$\geq$ constraint	≤ 0 variable	$A^\top y \,?\, c$
$x \,?\, 0$	≥ 0 variable	$\geq$ constraint	$y \,?\, 0$
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \ ?\ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0$$

Recap: Weak Duality

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$	
max $c^\top x$ subject to $Ax \,?\, b$, $x \,?\, 0$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \,?\, c$, $y \,?\, 0$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0$$

Recap: Weak Duality

$$\max\ (2,-1,3)x \qquad \text{(P)}$$
$$\text{s.t.} \begin{pmatrix}1 & 0 & -1\\ 0 & -2 & 1\\ 1 & 1 & 0\end{pmatrix} x \begin{matrix}\le\\ =\\ \ge\end{matrix} \begin{pmatrix}2\\ 1\\ -2\end{pmatrix}$$
$$x_1 \ge 0, x_2 \le 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$
$$\text{s.t.} \begin{pmatrix}1 & 0 & 1\\ 0 & -2 & 1\\ -1 & 1 & 0\end{pmatrix} y \begin{matrix}\ge\\ \le\\ =\end{matrix} \begin{pmatrix}2\\ -1\\ 3\end{pmatrix}$$
$$y_1 \ge 0, y_2 \text{ free}, y_3 \le 0$$

Weak Duality Theorem

if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D),

$$\implies c^T\bar{x} \le b^T\bar{y}$$

Recap: Weak Duality

$$\begin{array}{llcl} \max & (2,-1,3)x & & \text{(P)} \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x & \begin{matrix} \le \\ = \\ \ge \end{matrix} & \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \\ & x_1 \ge 0, x_2 \le 0, x_3 \text{ free} & & \end{array}$$

$$\begin{array}{llcl} \min & (2,1,-2)y & & \text{(D)} \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y & \begin{matrix} \ge \\ \le \\ = \end{matrix} & \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \\ & y_1 \ge 0, y_2 \text{ free}, y_3 \le 0 & & \end{array}$$

Weak Duality Theorem

if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D),

$$\implies \quad c^T\bar{x} \le b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then both $\bar{x}$ and $\bar{y}$ are optimal.

This Lecture: Strong Duality

(P_{max})			(P_{min})	
max $c^\top x$ subject to $Ax \mathbf{?}\, b$, $x \mathbf{?}\, \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbf{?}\, c$, $y \mathbf{?}\, \mathbb{0}$	
	$=$ constraint	free variable		
	$\geq$ constraint	≤ 0 variable		
	≥ 0 variable	$\geq$ constraint		
	free variable	$=$ constraint		
	≤ 0 variable	$\leq$ constraint		

Question

Can we always find feasible solutions $\bar{x}$ and $\bar{y}$ to a primal-dual pair, (P_{max}), (P_{min}), such that $c^T\bar{x} = b^T\bar{y}$?

This Lecture: Strong Duality

(P_{max})			(P_{min})
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, \mathbb{0}$	free variable	$=$ constraint	$y \,?\, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Question

Can we always find feasible solutions $\bar{x}$ and $\bar{y}$ to a primal-dual pair, (P_{max}), (P_{min}), such that $c^T\bar{x} = b^T\bar{y}$?

Strong Duality Theorem

If (P_{max}) has an optimal solution $\bar{x}$, then (P_{min}) has an optimal solution $\bar{y}$ such that $c^T\bar{x} = b^T\bar{y}$.

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

Let's assume (P) has an optimal solution.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

Let's assume (P) has an optimal solution.

$\longrightarrow$ 2-Phase Simplex terminates with an optimal basis B

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

Let's assume (P) has an optimal solution.

$\longrightarrow$ 2-Phase Simplex terminates with an optimal basis B (Why?)

$$\begin{aligned} \max\ & c^T x \qquad \text{(P)} \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \min\ & b^T y \qquad \text{(D)} \\ \text{s.t. } & A^T y \geq c \end{aligned}$$

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

Let's assume (P) has an optimal solution.

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Goal: Show that $\bar{y}$ is dual feasible.

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Note that B is an optimal basis $\longrightarrow$ $\bar{c} \leq \mathbb{0}$

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Equivalently, $A^T \bar{y} \geq c$,

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Strong Duality Theorem

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Let (P) and (D) be a primal-dual pair of LPs. If (P) has an optimal solution, then (D) has one, and their objective values equal.

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Fundamental Theorem of LP $\longrightarrow$ (P) has an optimal solution.

Subtly different version via previous results:

Strong Duality Theorem – Feasibility Version

Let (P) and (D) be primal-dual pair of LPs. If both are feasible, then both have optimal solutions of the same objective value.

Possible Outcomes of Primal-Dual Pair (P), (D)

(D) \ (P)	optimal solution	unbounded	infeasible
optimal solution			
unbounded			
infeasible			

Possible Outcomes of Primal-Dual Pair (P), (D)

(P) / (D)	optimal solution	unbounded	infeasible
optimal solution	possible ①		
unbounded			possible ⑥
infeasible		possible ⑧	

- ①, ⑥, and ⑧ many examples exist

Possible Outcomes of Primal-Dual Pair (P), (D)

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optimal solution	possible ①	impossible ②	
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  Suppose, for a contradiction, that (D) has an optimal solution $\bar{y}$.

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$$\max\ c^Tx \qquad \text{(P)}$$
$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

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Possible Outcomes of Primal-Dual Pair (P), (D)

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  Similar arguments apply to ④ and ⑤
- ③, ⑦ follow directly from Strong Duality
- I'll leave ⑨ for you to do as an exercise!

$$\max\ c^Tx \qquad \text{(P)}$$
$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

$$\min\ b^Ty \qquad \text{(D)}$$
$$\text{s.t. } A^Ty \geq c$$

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Strong Duality Theorem

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Module 4: Duality Theory (Geometry of Duality)

Recap: Strong Duality

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

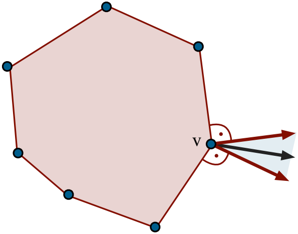
Strong Duality Theorem

For the above primal-dual pair of LPs, (P) and (D), if (P) has an optimal solution, then (D) has one and their objective values equal.

Recap: The Geometry of an LP

In Module 2, we saw that

- The feasible region of an LP is a polyhedron.
- Basic solutions correspond to extreme points of this polyhedron.



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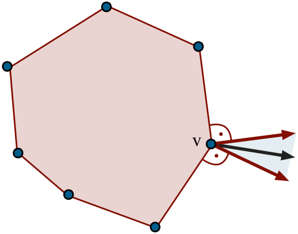

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When is an extreme point optimal?

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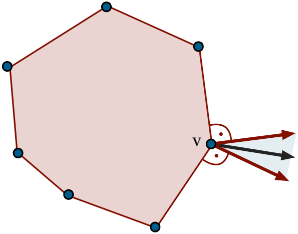

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Module 2 and strong duality told us that Simplex computes

- a basic solution (if it exists), and
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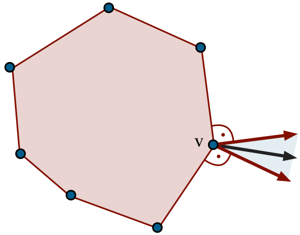

Question

When is an extreme point optimal?

Module 2 and strong duality told us that Simplex computes

- a basic solution (if it exists), and
- a certificate of optimality.

Today we will investigate these certificates using geometry.

Revisiting Weak Duality

We can rewrite (P) using slack variables s:

$$\begin{aligned} \max\ & c^T x & \text{(P')} \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \max\ & c^T x & \text{(P)} \\ \text{s.t. } & Ax \leq b \end{aligned}$$

$$\begin{aligned} \min\ & b^T y & \text{(D)} \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned}$$

Revisiting Weak Duality

We can rewrite (P) using slack variables s:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

Note:

- (x, s) feasible for (P') $\longrightarrow$ x feasible for (P)

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality

We can rewrite (P) using slack variables s:

$$\max\ c^T x \qquad \text{(P')}$$
$$\text{s.t. } Ax + s = b$$
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$$\max\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \leq b$$

$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y = c$$
$$y \geq \mathbb{0}$$

Note:

- (x, s) feasible for (P') $\longrightarrow$ x feasible for (P)
- x feasible for (P) $\longrightarrow (x, b - Ax)$ feasible for (P')

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

$$\longrightarrow \quad (\bar{x}, b - A\bar{x}) \text{ feasible for (P')}$$

$$\begin{aligned} \max\ & c^T x & \text{(P)} \\ \text{s.t. } & Ax \leq b \end{aligned}$$

$$\begin{aligned} \max\ & c^T x & \text{(P')} \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \min\ & b^T y & \text{(D)} \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned}$$

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

$$\longrightarrow \quad (\bar{x}, \underbrace{b - A\bar{x}}_{\bar{s}}) \text{ feasible for (P')}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \ge \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \ge \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

$$\longrightarrow \quad (\bar{x}, \underbrace{b - A\bar{x}}_{\bar{s}}) \text{ feasible for (P')}$$

Recall the Weak Duality proof:

$$\begin{aligned} \bar{y}^T b &= \bar{y}^T (A\bar{x} + \bar{s}) \\ &= (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} \end{aligned}$$

$$\begin{aligned} \max\ & c^T x & \text{(P)} \\ \text{s.t. } & Ax \le b \end{aligned}$$

$$\begin{aligned} \max\ & c^T x & \text{(P')} \\ \text{s.t. } & Ax + s = b \\ & s \ge \mathbb{0} \end{aligned}$$

$$\begin{aligned} \min\ & b^T y & \text{(D)} \\ \text{s.t. } & A^T y = c \\ & y \ge \mathbb{0} \end{aligned}$$

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

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$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \ge \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \ge \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

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Strong Duality tells us that:

$$\bar{x},\ \bar{y} \text{ both optimal} \quad \Longleftrightarrow \quad c^T \bar{x} = \bar{y}^T b$$

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax + s = b \\ &\quad s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} &\min\ b^T y \\ &\text{s.t. } A^T y = c \\ &\quad y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

$$\longrightarrow \quad (\bar{x}, \underbrace{b - A\bar{x}}_{\bar{s}}) \text{ feasible for (P')}$$

Recall the Weak Duality proof:

$$\begin{aligned} \bar{y}^T b &= \bar{y}^T (A\bar{x} + \bar{s}) \\ &= (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} \\ &= c^T \bar{x} + \bar{y}^T \bar{s} \end{aligned}$$

Strong Duality tells us that:

$$\begin{aligned} \bar{x},\ \bar{y} \text{ both optimal} \quad &\Longleftrightarrow \quad c^T \bar{x} = \bar{y}^T b \\ &\Longleftrightarrow \quad \bar{y}^T \bar{s} = 0 \end{aligned}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \ge \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \ge \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality

Recall the Weak Duality proof:

$$\begin{aligned}\bar{y}^T b &= \bar{y}^T (A\bar{x} + \bar{s}) \\ &= (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} \\ &= c^T \bar{x} + \bar{y}^T \bar{s}\end{aligned}$$

$$0 = \bar{y}^T \bar{s} = \sum_{i=1}^{m} \bar{y}_i \bar{s}_i \qquad (\star)$$

$$\max c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \le b$$

$$\max c^T x \qquad \text{(P')}$$
$$\text{s.t. } Ax + s = b$$
$$s \ge \mathbb{0}$$

$$\min b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y = c$$
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Revisiting Weak Duality

Recall the Weak Duality proof:

$$\begin{aligned}\bar{y}^T b &= \bar{y}^T(A\bar{x} + \bar{s}) \\ &= (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} \\ &= c^T \bar{x} + \bar{y}^T \bar{s}\end{aligned}$$

$$0 = \bar{y}^T \bar{s} = \sum_{i=1}^{m} \bar{y}_i \bar{s}_i \qquad (\star)$$

By feasibility, $\bar{y} \geq \mathbb{0}$ and $\bar{s} \geq \mathbb{0}$

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax + s = b \\ &\quad s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} &\min\ b^T y \\ &\text{s.t. } A^T y = c \\ &\quad y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality

Recall the Weak Duality proof:

$$\begin{aligned} \bar{y}^T b &= \bar{y}^T(A\bar{x} + \bar{s}) \\ &= (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} \\ &= c^T \bar{x} + \bar{y}^T \bar{s} \end{aligned}$$

$$0 = \bar{y}^T \bar{s} = \sum_{i=1}^{m} \bar{y}_i \bar{s}_i \qquad (\star)$$

By feasibility, $\bar{y} \geq \mathbb{0}$ and $\bar{s} \geq \mathbb{0}$ and hence

$(\star)$ holds if and only if $\bar{y}_i = 0$ or $\bar{s}_i = 0$,

for every $1 \leq i \leq m$.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality – Recap

Given: $\bar{x}$ and $\bar{y}$ feasible solutions for (P) and (D)

$$\max\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \leq b$$

$$\max\ c^T x \qquad \text{(P')}$$
$$\text{s.t. } Ax + s = b$$
$$s \geq \mathbb{0}$$

$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y = c$$
$$y \geq \mathbb{0}$$

Revisiting Weak Duality – Recap

Given: $\bar{x}$ and $\bar{y}$ feasible solutions for (P) and (D)

Define: $\bar{s} = b - A\bar{x}$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \ge \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \ge \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality – Recap

Given: $\bar{x}$ and $\bar{y}$ feasible solutions for (P) and (D)

Define: $\bar{s} = b - A\bar{x}$

Then:

$\bar{x}$ and $\bar{y}$ optimal $\iff$ $\bar{y}_i = 0$ or $\bar{s}_i = 0$

for all $1 \leq i \leq m$.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality – Recap

Given: $\bar{x}$ and $\bar{y}$ feasible solutions for (P) and (D)

Define: $\bar{s} = b - A\bar{x}$

Then:

$\bar{x}$ and $\bar{y}$ optimal $\iff$ $\underbrace{\bar{y}_i = 0 \text{ or } \bar{s}_i = 0}_{(\star)}$

for all $1 \leq i \leq m$.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

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$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality – Recap

Given: $\bar{x}$ and $\bar{y}$ feasible solutions for (P) and (D)

Define: $\bar{s} = b - A\bar{x}$

Then:

$\bar{x}$ and $\bar{y}$ optimal $\iff$ $\underbrace{\bar{y}_i = 0 \text{ or } \bar{s}_i = 0}_{(\star)}$

for all $1 \leq i \leq m$. We can rephrase $(\star)$ equivalently as

$\bar{y}_i = 0$ or ith constraint of (P) holds with equality .

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality – Recap

Given: $\bar{x}$ and $\bar{y}$ feasible solutions for (P) and (D)

Define: $\bar{s} = b - A\bar{x}$

Then:

$$\bar{x} \text{ and } \bar{y} \text{ optimal} \quad \Longleftrightarrow \quad \underbrace{\bar{y}_i = 0 \text{ or } \bar{s}_i = 0}_{(\star)}$$

for all $1 \leq i \leq m$. We can rephrase $(\star)$ equivalently as

$\bar{y}_i = 0$ or ith constraint of (P) holds with equality (is tight).

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Complementary Slackness – Special Case

Let $\bar{x}$ and $\bar{y}$ be feasible for (P) and (D).

Then $\bar{x}$ and $\bar{y}$ are optimal if and only if

(i) $\bar{y}_i = 0$, or

(ii) the ith constraint of (P) is tight for $\bar{x}$,

for every row index i.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \tag{P}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \tag{P'}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \tag{D}$$

Complementary Slackness Conditions – Example

Consider the following LP:

$$\max\ (5,3,5)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

Complementary Slackness Conditions – Example

Consider the following LP:

$$\max\ (5,3,5)x \qquad \text{(P)}$$
$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

Its dual is:

$$\min\ (2,4,-1)y \qquad \text{(D)}$$
$$\text{s.t.}\ \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} y = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$$
$$y \geq \mathbb{0}$$

Complementary Slackness Conditions – Example

Consider the following LP:

$$\max\ (5,3,5)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

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$$\text{s.t.}\ \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} y = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$$

$$y \ge \mathbb{0}$$

Claim

$\bar{x} = (1,-1,1)^T$ and $\bar{y} = (0,2,1)^T$ are optimal!

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Consider the following LP:

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$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

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$$\text{s.t.}\ \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} y = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$$
$$y \ge \mathbb{0}$$

Claim

$\bar{x} = (1,-1,1)^T$ and $\bar{y} = (0,2,1)^T$ are optimal!

Complementary Slackness

Feasible solutions $\bar{x}$ and $\bar{y}$ for (P) and (D) are optimal if and only if

$\bar{y}_i = 0$ or the ith primal constraint is tight for $\bar{x}$, for all row indices i.

Complementary Slackness Conditions – Example

Consider the following LP:

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$$y \geq \mathbb{0}$$

Claim

$\bar{x} = (1,-1,1)^T$ and $\bar{y} = (0,2,1)^T$ are optimal!

Complementary Slackness

Feasible solutions $\bar{x}$ and $\bar{y}$ for (P) and (D) are optimal if and only if

$\bar{y}_i = 0$ or the ith primal constraint is tight for $\bar{x}$, for all row indices i.

It is easy to check if $\bar{x}$ and $\bar{y}$ are feasible.

Complementary Slackness Conditions – Example

Consider the following LP:

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Feasible solutions $\bar{x}$ and $\bar{y}$ for (P) and (D) are optimal if and only if

$\bar{y}_i = 0$ or the ith primal constraint is tight for $\bar{x}$, for all row indices i.

It is easy to check if $\bar{x}$ and $\bar{y}$ are feasible.

(i) $\bar{y}_1 = 0$ or $(1,2,-1)\bar{x} = 2$

Complementary Slackness Conditions – Example

Consider the following LP:

$$\max\ (5,3,5)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

Its dual is:

$$\min\ (2,4,-1)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} y = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$$

$$y \geq \mathbb{0}$$

Claim

$\bar{x} = (1,-1,1)^T$ and $\bar{y} = (0,2,1)^T$ are optimal!

Complementary Slackness

Feasible solutions $\bar{x}$ and $\bar{y}$ for (P) and (D) are optimal if and only if

$\bar{y}_i = 0$ or the ith primal constraint is tight for $\bar{x}$, for all row indices i.

It is easy to check if $\bar{x}$ and $\bar{y}$ are feasible.

(i) $\bar{y}_1 = 0$ or $(1,2,-1)\bar{x} = 2$

(ii) $\bar{y}_2 = 0$ or $(3,1,2)\bar{x} = 4$

Complementary Slackness Conditions – Example

Consider the following LP:

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$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

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Claim

$\bar{x} = (1,-1,1)^T$ and $\bar{y} = (0,2,1)^T$ are optimal!

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Feasible solutions $\bar{x}$ and $\bar{y}$ for (P) and (D) are optimal if and only if

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It is easy to check if $\bar{x}$ and $\bar{y}$ are feasible.

(i) $\bar{y}_1 = 0$ or $(1,2,-1)\bar{x} = 2$

(ii) $\bar{y}_2 = 0$ or $(3,1,2)\bar{x} = 4$

(iii) $\bar{y}_3 = 0$ or $(-1,1,1)\bar{x} = -1$

Complementary Slackness Conditions – Example

Consider the following LP:

$$\max\ (5,3,5)x \qquad \text{(P)}$$
$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

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$$y \geq \mathbb{0}$$

Claim

$\bar{x} = (1,-1,1)^T$ and $\bar{y} = (0,2,1)^T$ are optimal!

Complementary Slackness

Feasible solutions $\bar{x}$ and $\bar{y}$ for (P) and (D) are optimal if and only if

$\bar{y}_i = 0$ or the ith primal constraint is tight for $\bar{x}$, for all row indices i.

It is easy to check if $\bar{x}$ and $\bar{y}$ are feasible.

(i) $\bar{y}_1 = 0$ or $(1,2,-1)\bar{x} = 2$

(ii) $\bar{y}_2 = 0$ or $(3,1,2)\bar{x} = 4$

(iii) $\bar{y}_3 = 0$ or $(-1,1,1)\bar{x} = -1$

$\longrightarrow$ $\bar{x}$ and $\bar{y}$ are optimal!

General Complementary Slackness

$(\mathrm{P}_{\max})$					
max	$c^\top x$	$\leq$ constraint	≥ 0 variable	min	$b^\top y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \, ? \, b$	$\geq$ constraint	≤ 0 variable		$A^\top y \, ? \, c$
	$x \, ? \, \mathbb{0}$	≥ 0 variable	$\geq$ constraint		$y \, ? \, \mathbb{0}$
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

Suppose: $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$ are a pair of primal and dual LPs according to the above table

General Complementary Slackness

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, \mathbb{0}$	free variable	$=$ constraint	$y \,?\, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Suppose: $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$ are a pair of primal and dual LPs according to the above table, with feasible solutions $\bar{x}$, and $\bar{y}$

General Complementary Slackness

(P_{max})			(P_{min})		
max $c^\top x$		$\leq$ constraint	≥ 0 variable	min $b^\top y$	
subject to		$=$ constraint	free variable	subject to	
	$Ax\ ?\ b$	$\geq$ constraint	≤ 0 variable		$A^\top y\ ?\ c$
	$x\ ?\ \mathbb{0}$	≥ 0 variable	$\geq$ constraint		$y\ ?\ \mathbb{0}$
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

Suppose: (P_{max}) and (P_{min}) are a pair of primal and dual LPs according to the above table, with feasible solutions $\bar{x}$, and $\bar{y}$

$\bar{x}$ and $\bar{y}$ satisfy the complementary slackness conditions if ...

for all variables x_j of (P_{max}):

(i) $\bar{x}_j = 0$, or

(ii) jth constraint of (P_{min}) is satisfied with equality for $\bar{y}$

General Complementary Slackness

(P_{max})			(P_{min})
max $c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} \mathbb{0}$	$\leq$ constraint $=$ constraint $\geq$ constraint ≥ 0 variable free variable ≤ 0 variable	≥ 0 variable free variable ≤ 0 variable $\geq$ constraint $=$ constraint $\leq$ constraint	min $b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} \mathbb{0}$

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General Complementary Slackness

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Complementary Slackness Theorem

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\bar{x}$ and $\bar{y}$ be feasible solutions. Then these solutions are optimal if and only if the CS conditions hold.

General CS Conditions – Example

($\mathrm{P}_{\max}$)			($\mathrm{P}_{\min}$)
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \ ? \ b$	≥ 0 variable	$\geq$ constraint	$A^\top y \ ? \ c$
$x \ ? \ \mathbb{0}$	free variable	$=$ constraint	$y \ ? \ \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Consider the following LP...

$$\max \ (-2,-1,0)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$x_1 \leq 0, x_2 \geq 0$$

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$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
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Check: $\bar{x} = (-1,0,3)^T$ and $\bar{y} = (-1,1)^T$ are feasible for (P) and (D).

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Are they also optimal?

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$$\text{s.t.}\ \begin{pmatrix}1 & -1\\ 3 & 4\\ 2 & 2\end{pmatrix} y \begin{matrix}\leq\\ \geq\\ =\end{matrix} \begin{pmatrix}-2\\-1\\0\end{pmatrix}$$
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Complementary Slackness Theorem

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\bar{x}$ and $\bar{y}$ be feasible solutions. Then these solutions are optimal if and only if the CS conditions hold.

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Claim

$\bar{x} = (-1,0,3)^T$ and $\bar{y} = (-1,1)^T$ are optimal

Primal conditions:

(i) $\bar{x}_1 = 0$ or the first (D) constraint is tight for $\bar{y}$.

(ii) $\bar{x}_2 = 0$ or the second (D) constraint is tight for $\bar{y}$.

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Dual conditions:

(i) $\bar{y}_1 = 0$ or the first (P) constraint is tight for $\bar{x}$.

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Complementary Slackness – Geometry

Complementary Slackness Theorem

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\bar{x}$ and $\bar{y}$ be feasible solutions. Then these solutions are optimal if and only if the CS conditions hold.

Will now see a geometric interpretation of this theorem!

Complementary Slackness – Geometry

Complementary Slackness Theorem

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Will now see a geometric interpretation of this theorem!

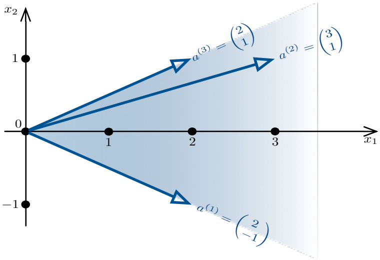
But some basics first!

Geometry – Cones of Vectors

Definition

Let $a^{(1)}, \ldots, a^{(k)}$ be vectors in $\mathbb{R}^n$. The cone generated by these vectors is given by

$$C = \{\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} \ : \ \lambda \geq \mathbb{0}\}$$



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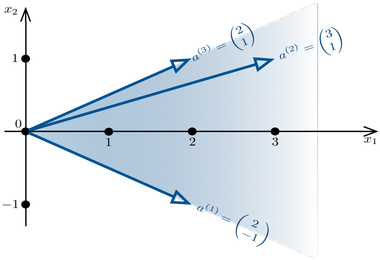
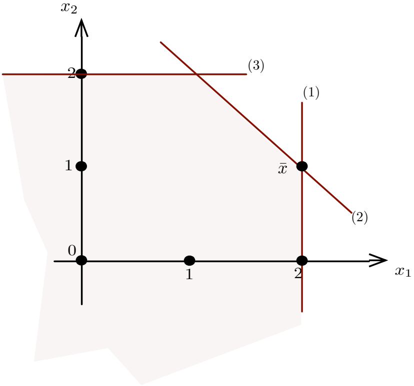


Example: The cone generated by $a^{(1)}, a^{(2)}$ and $a^{(3)}$ is the blue-shaded area.

Geometry – Cone of Tight Constraints

Consider the following polyhedron:

$$P = \{x \in \mathbb{R}^2 \ : \ \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A} x \leq \underbrace{\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}}_{b} \}$$



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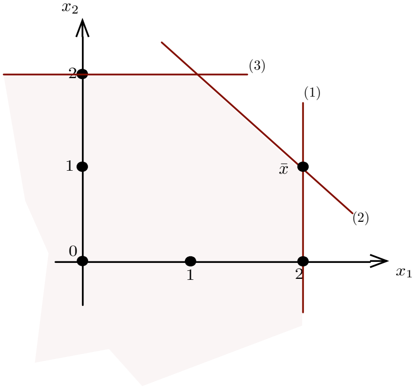

Consider: $\bar{x} = (2,1)^T$

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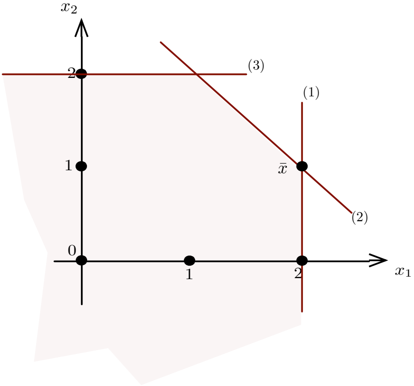

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(i) $\bar{x} \in P \longrightarrow$ Check!

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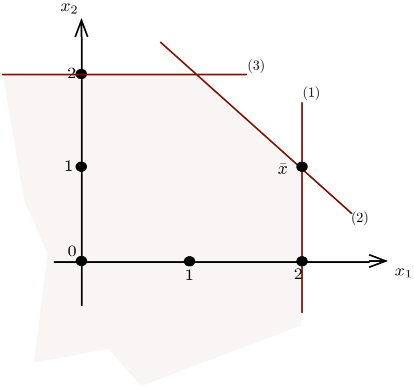

Consider: $\bar{x} = (2, 1)^T$

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(ii) Tight constraints:

$$\text{row}_1(A)\bar{x} = b_1$$
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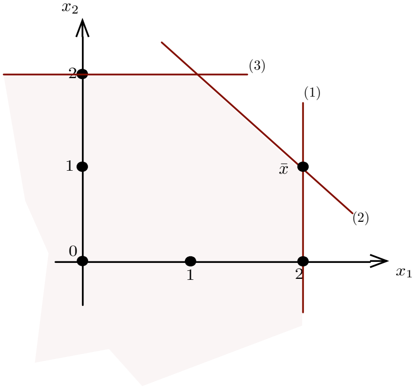

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$$\text{row}_1(A)\bar{x} = b_1 \quad \longrightarrow \quad (1,0)\bar{x} = 2$$
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Cone of tight constraints:

Cone generated by rows of tight constraints

Geometry – Cone of Tight Constraints

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$$(1,0)\bar{x} = 2 \quad (1)$$

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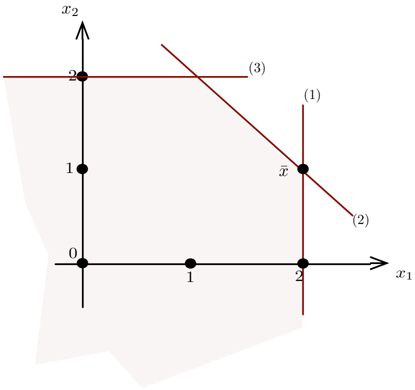

Geometry – Cone of Tight Constraints

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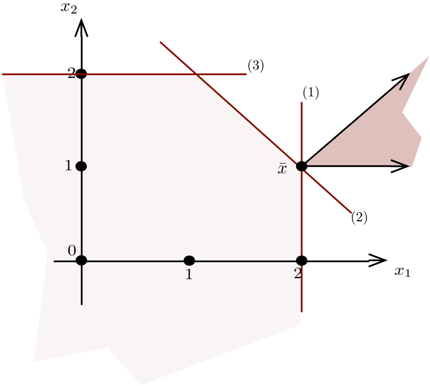
$$\{\lambda_1(1,0)^T + \lambda_2(1,1)^T \,:\, \lambda_1, \lambda_2 \geq 0\}$$



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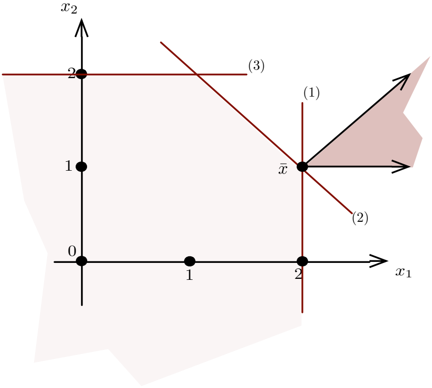
Consider an LP of the form

$$\max\{c^T x \,:\, Ax \leq b\}$$

and a feasible solution $\bar{x}$.



Geometry – Cone of Tight Constraints

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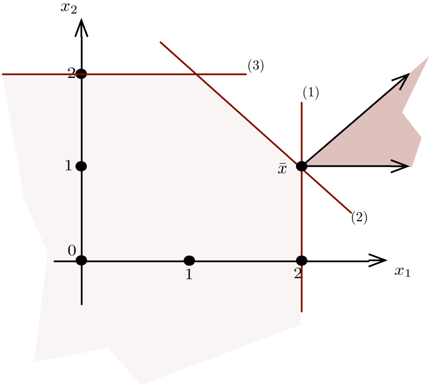

The cone of tight constraints at $\bar{x}$ is the cone generated by the rows of A corresponding to tight constraints at $\bar{x}$.

Theorem

Let $\bar{x}$ be a feasible solution to

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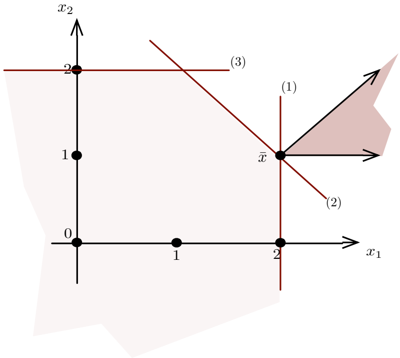
Then $\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.



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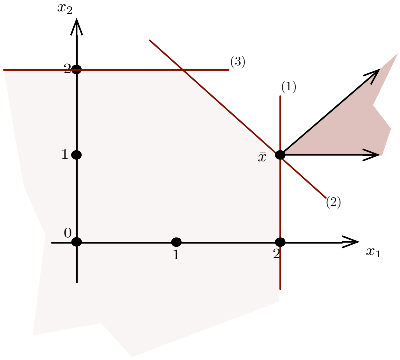

Example: Consider the LP

$$\max\{(3/2, 1/2)x \,:\, x \in P\}$$

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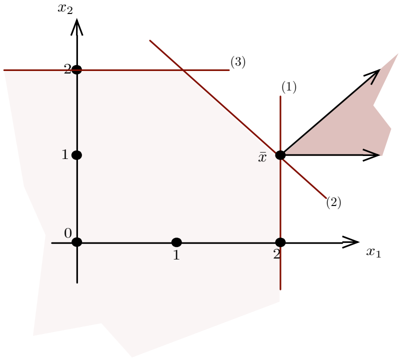

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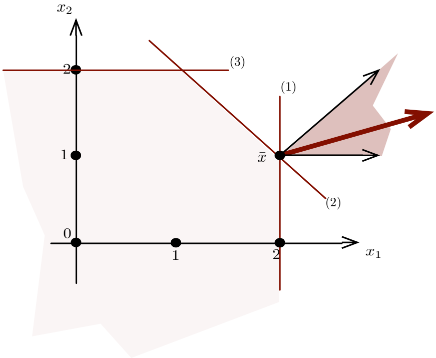

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Proving the "if" direction of the above theorem amounts to

(i) finding a feasible solution $\bar{y}$ to the dual of $(\star)$, and

(ii) showing that $\bar{x}$ and $\bar{y}$ satisfy the CS conditions!

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The above theorem follows from CS Theorem!

Geometric Optimality – Towards a Proof

If we write out the LP:

$$\max\ (3/2, 1/2)x \qquad\qquad (\star)$$

$$\text{s.t.}\ \begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{pmatrix} x \le \begin{pmatrix}2\\ 3\\ 2\end{pmatrix}$$

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CS Theorem $\longrightarrow$ $(\bar{x}, \bar{y})$ optimal!

Suppose $\bar{x}$ is a solution to (P), and let $J(\bar{x})$ be the indices of tight constraints for $\bar{x}$.

$$\max\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \leq b$$

$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y = c$$
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(x, y) satisfy CS Conditions if for all variables y_i of (D):

$$y_i = 0 \quad \text{or} \quad \text{row}_i(A)x = b_i \qquad (\star)$$

Suppose $\bar{x}$ is a solution to (P), and let $J(\bar{x})$ be the indices of tight constraints for $\bar{x}$. i.e.,

$$\text{row}_i(A)\bar{x} = b_i$$

for $i \in J(\bar{x})$ and

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Suppose c is in the cone of tight constraints at $\bar{x}$, and thus

$$c = \sum_{i \in J(\bar{x})} \lambda_i \mathsf{row}_i(A)^T$$

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Hence: $(\bar{x}, \bar{y})$ are optimal!

Wrapping up...

We almost proved:

Theorem

Let $\bar{x}$ be a feasible solution to

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Then $\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.

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CS Theorem $\longrightarrow$ there is a feasible dual solution $\bar{y}$ that, together with $\bar{x}$, satisfies CS conditions.

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CS Theorem $\longrightarrow$ there is a feasible dual solution $\bar{y}$ that, together with $\bar{x}$, satisfies CS conditions.

We can use CS conditions and $\bar{y}$ to show that c lies in cone of tight constraints for $\bar{x}$. This is an exercise!

Recap

Given a feasible solution $\bar{x}$ to

$$\max\{c^T x \,:\, Ax \leq b\}$$

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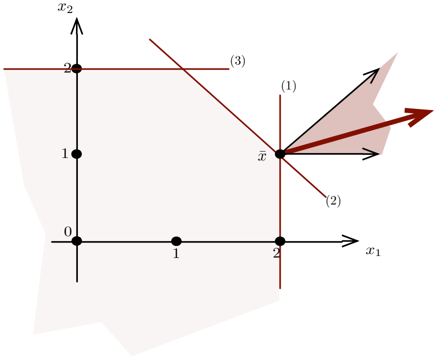

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Recap

Given a feasible solution $\bar{x}$ to

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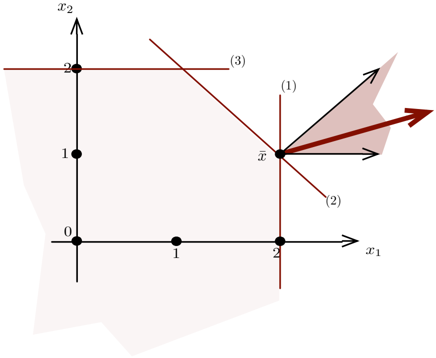
$\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.

This provides a nice geometric view of optimality certificates



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Module 5: Integer Programs (IP versus LP)

LP versus IP

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Can solve very large instances	

LP versus IP

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Algorithms exist that are guaranteed to be fast	

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Remark

We cannot PROVE an algorithm that is guaranteed to be fast does not exist, but we can show that it is "highly unlikely".

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Linear programming	Integer programming
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Let us look at an example...

A Bad Example

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Proposition

The following IP,

$$\boxed{\begin{array}{l} \max \quad x_1 - \sqrt{2}x_2 \\ \text{s.t.} \\ \qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ \qquad x_1, x_2 \text{ integer} \end{array}}$$

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Integer Programming can, in principle, be reduced to Linear Programming.

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This will NOT give us a practical procedure to solve IPs,

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This will NOT give us a practical procedure to solve IPs,

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Definition

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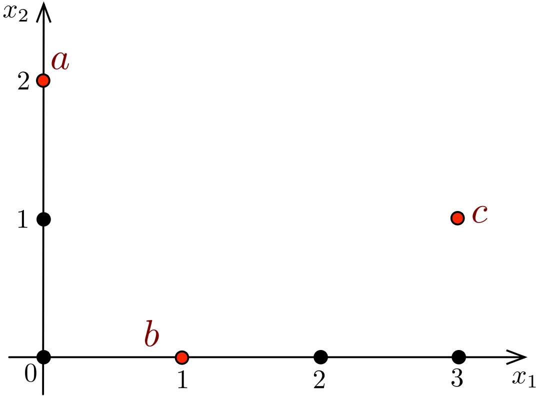

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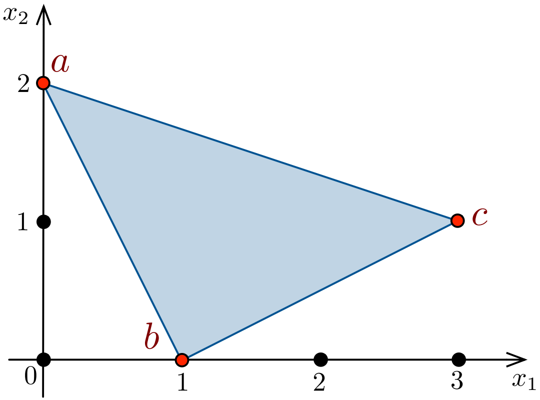

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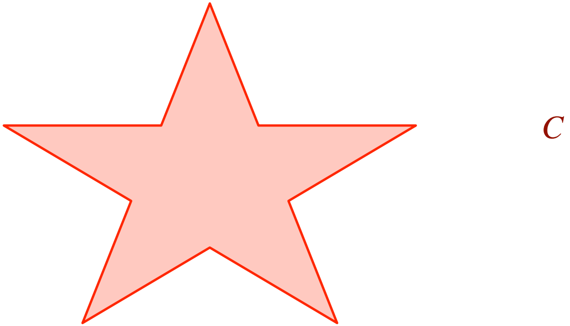

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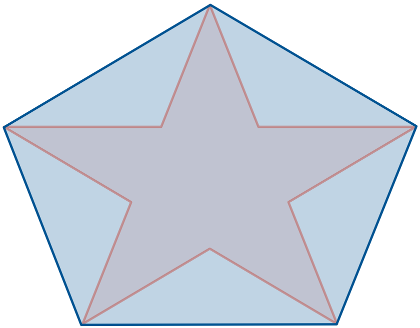

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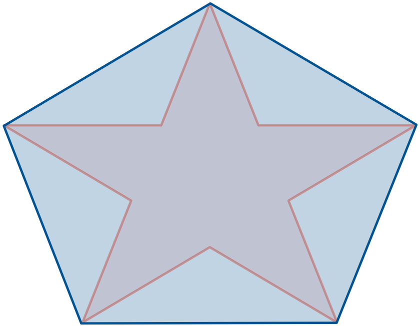

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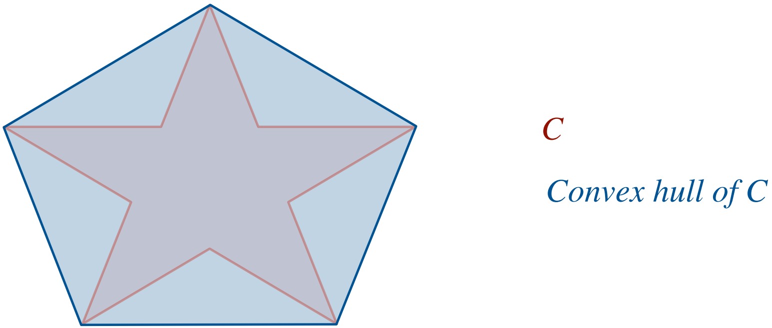

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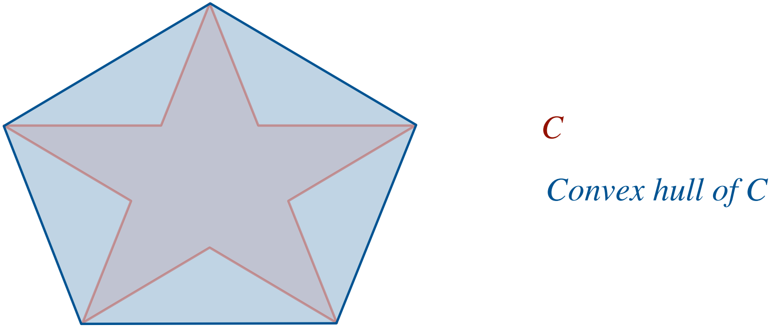

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➡ The notion of a convex hull is well defined.

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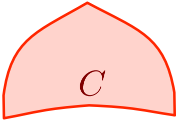

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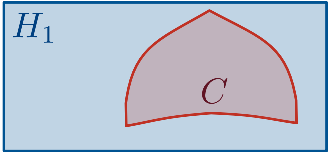

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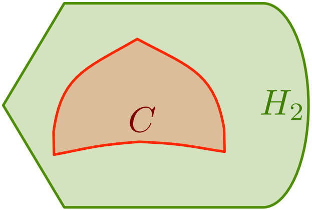

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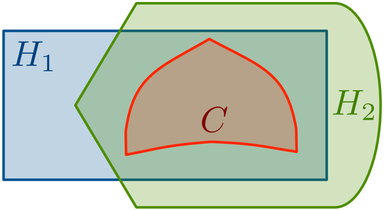

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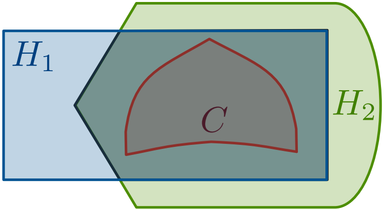

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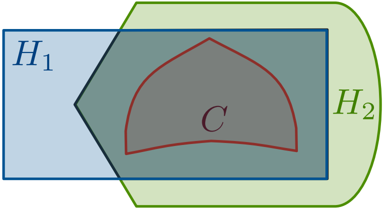

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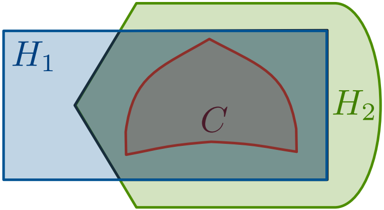

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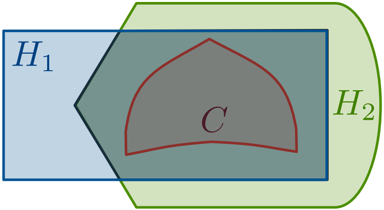

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However, $H_1 \cap H_2$ is smaller than both H_1 and H_2. This is a contradiction.

Convex Hulls and Integer Programs

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$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$

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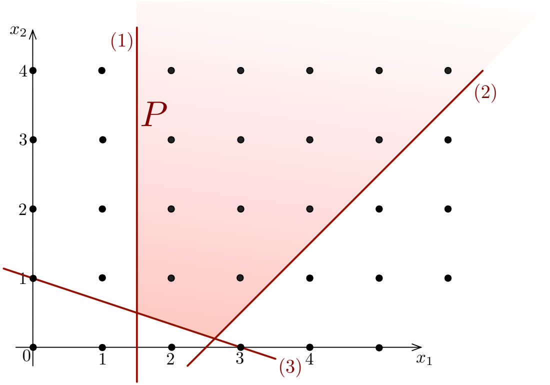

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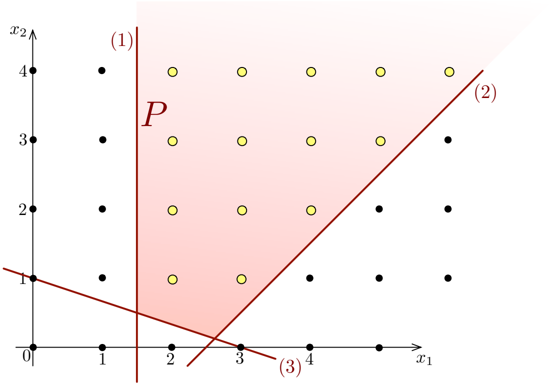

Integer points in P

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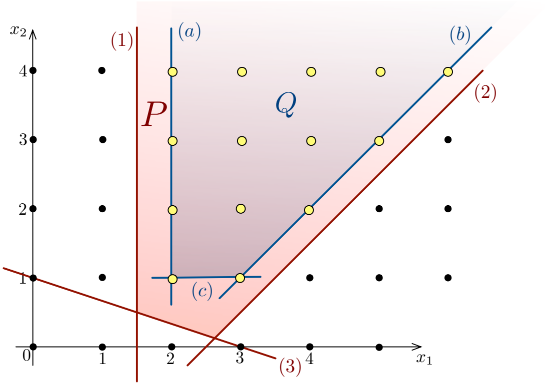

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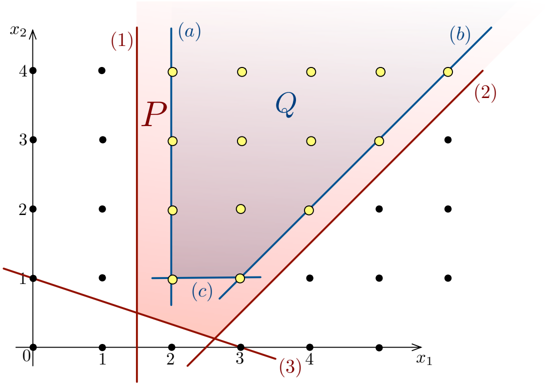

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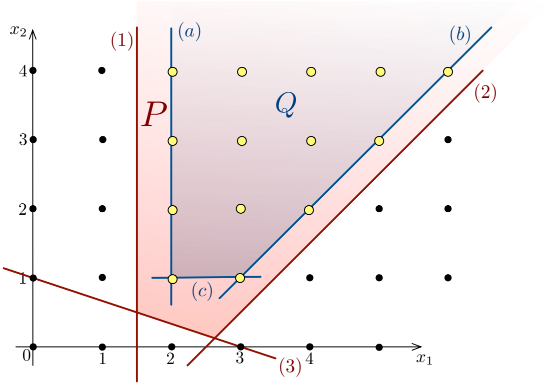

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POLYHEDRON

Meyer's Theorem

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<u>Goal</u>: Use Meyer's theorem to reduce the problem of solving *Integer Programs*, to the problem of solving *Linear Program*.

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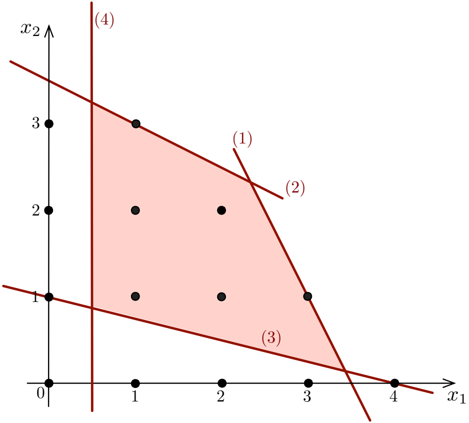

Feasible region of the LP relaxation of (IP)

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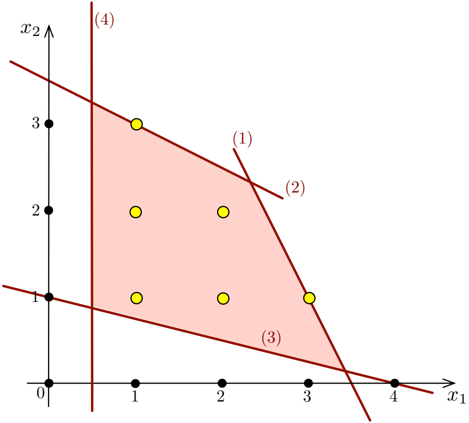

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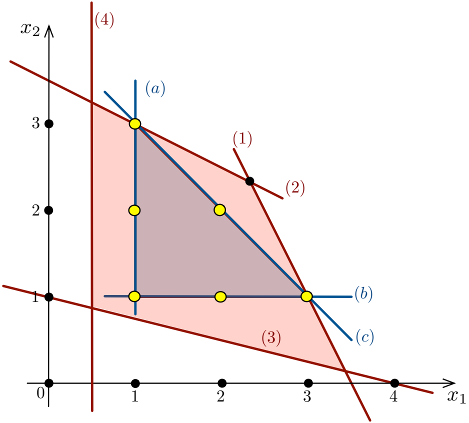

The convex hull of the feasible region of (IP)

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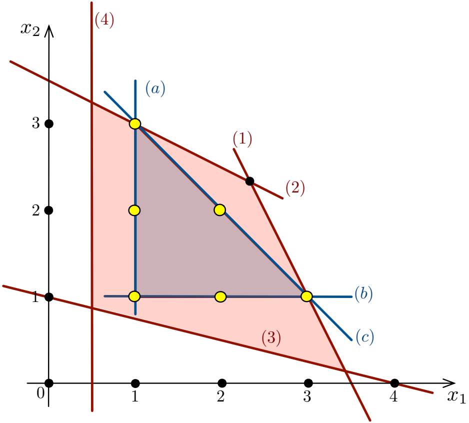

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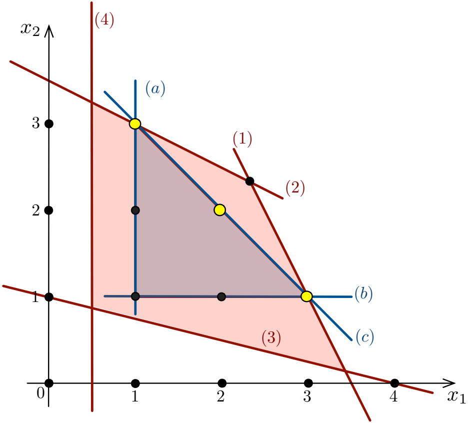

Optimal solutions of (IP)

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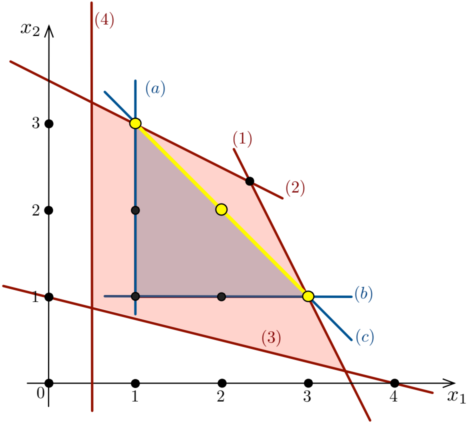

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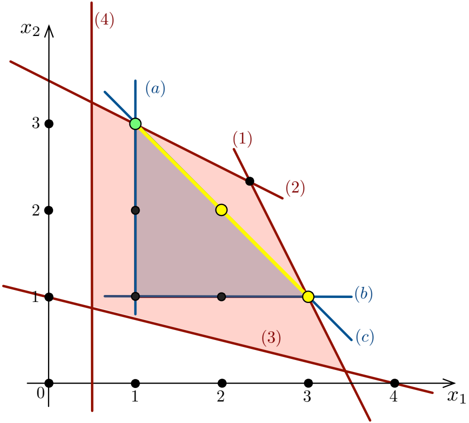

Extreme optimal solution of (LP)

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Step 1. Compute A', b'.

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$$\max\{c^\top x : A'x \leq b'\} \qquad \text{(LP)}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

Remark

This is NOT a practical way to solve an Integer Program.

WHY NOT?

- We do not know how to compute A', b', and

$$\max\{c^\top x : Ax \le b, x \text{ integer}\} \tag{IP}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \le b'\}$.

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Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

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This is NOT a practical way to solve an Integer Program.

WHY NOT?

- We do not know how to compute A', b', and
- A', b' can be MUCH more complicated than A, b.

Question

How do we fix these problems?

Question

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Idea

Construct an approximation of the convex hull of the solutions of (IP).

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Recap

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- Integer Programs are much harder to solve than Linear Programs.

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Construct an approximation of the convex hull of the solutions of (IP).

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- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.

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- We defined the notion of convex hulls.

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- Integer Programs are much harder to solve than Linear Programs.
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- We defined the notion of convex hulls.
- The convex hull of the integer points in a rational polyhedron is a polyhedron.
- Integer programming reduces to Linear programming,
  but it is NOT a practical reduction.

Module 5: Integer Programs (Cutting Planes)

Overview

In this lecture, we will:

Investigate a class of algorithms known as cutting planes.

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We restrict ourselves to pure Integer Programs.

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$$
\begin{aligned}
\max\ & \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
\text{s. t.}\ & \\
& \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
& x \ge \mathbf{0},\ x \text{ integer}
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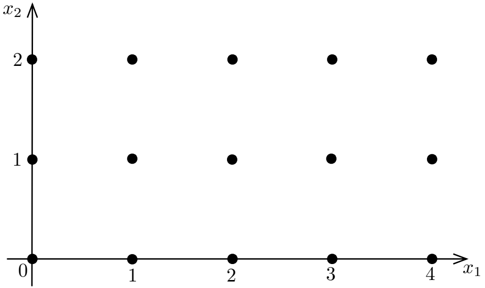

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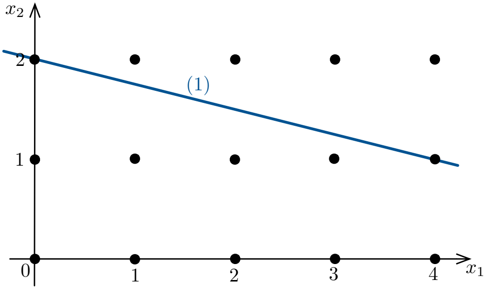

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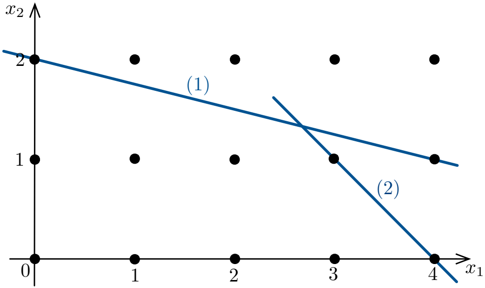

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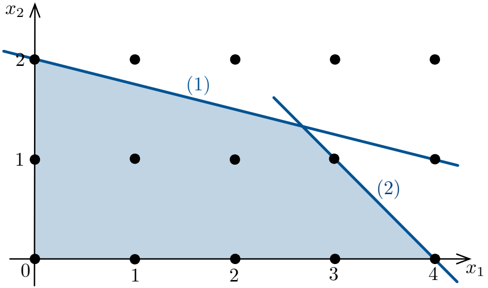

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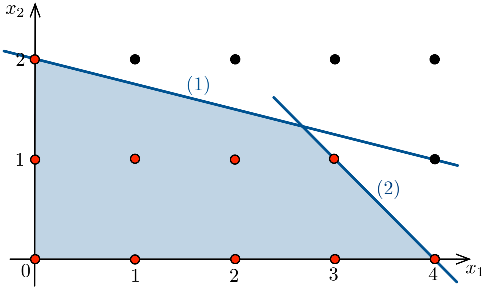

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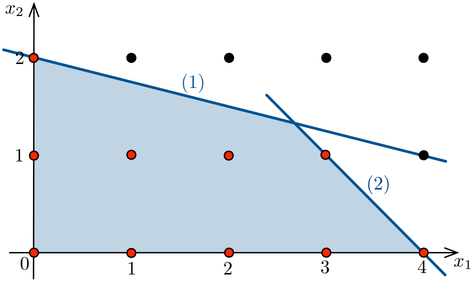

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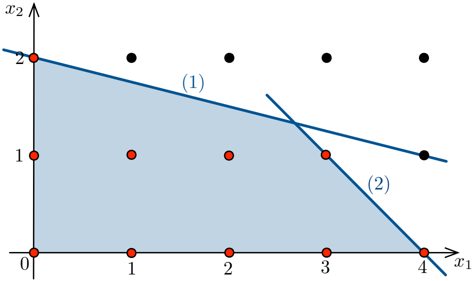

Idea

Solve the LP relaxation instead of the original IP.

$$\max \; \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

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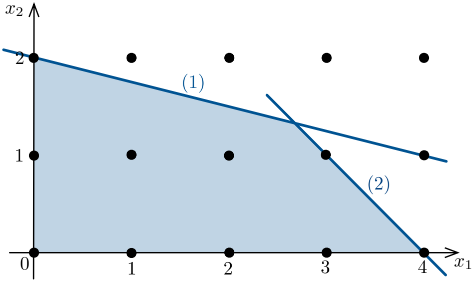

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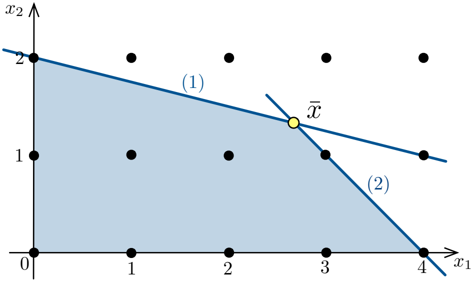

Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal.

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

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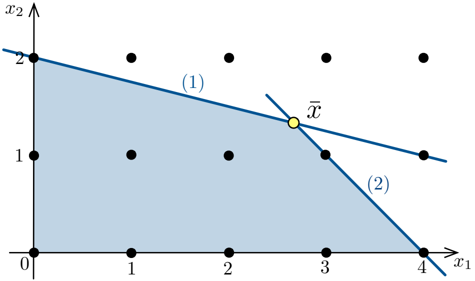

Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. Not integer!

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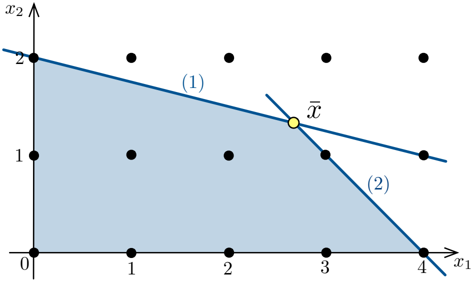

Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. Not integer!

We now search for a constraint $\alpha^\top x \le \beta$ that

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$
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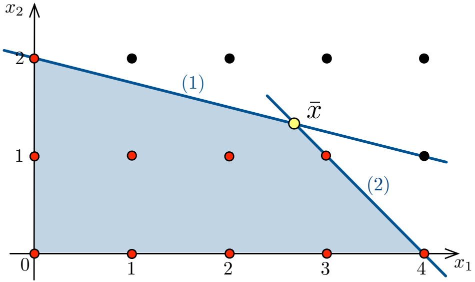

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- is satisfied for all feasible solutions to the IP, and

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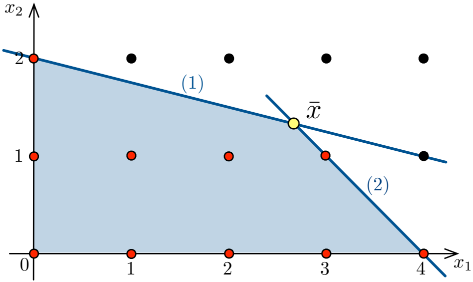

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We now search for a constraint $\alpha^\top x \leq \beta$ that

- is satisfied for all feasible solutions to the IP, and
- is not satisfied for $\bar{x}$.

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$
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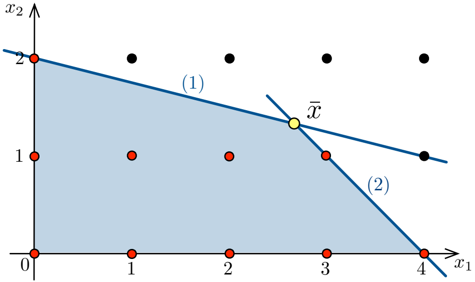

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We will call this constraint a cutting plane for $\bar{x}$.

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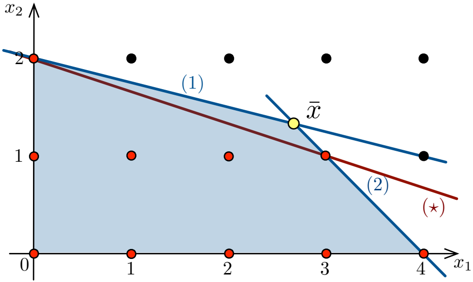

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Example:

$$x_1 + 3x_2 \leq 6. \qquad (\star)$$

After adding $(\star)$ to our relaxation, we get

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$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

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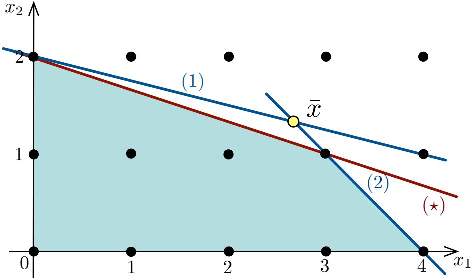

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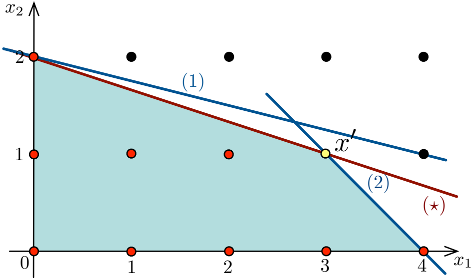

Using Simplex, we get: $x' = (3, 1)^\top$ is optimal.

After adding $(\star)$ to our relaxation, we get

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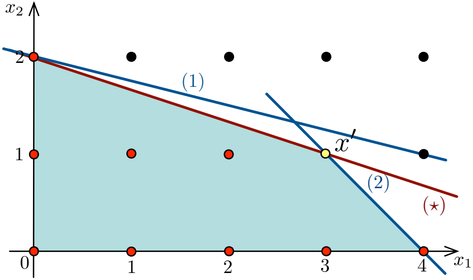

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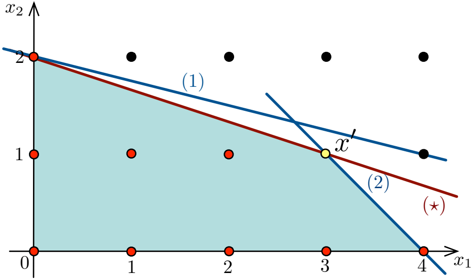
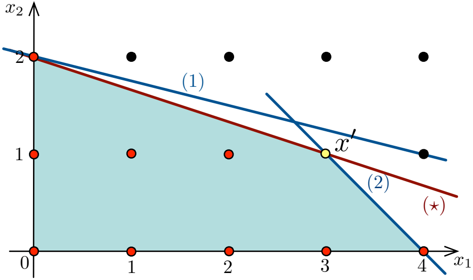


Using Simplex, we get: $x' = (3, 1)^\top$ is optimal. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

After adding $(\star)$ to our relaxation, we get

$$\begin{array}{l} \max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \quad \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \\ \qquad x \geq \mathbf{0} \end{array}$$



Using Simplex, we get: $x' = (3, 1)^\top$ is optimal. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

We have now solved our first IP.

Cutting Plane Scheme

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$$\max \left\{ c^\top x : Ax \leq b, x \text{ integer} \right\} \qquad \text{(IP)}$$

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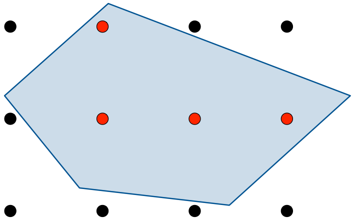


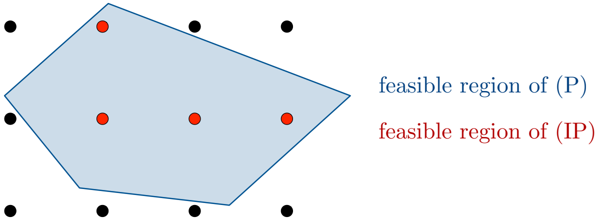
feasible region of (P)

feasible region of (IP)

- Let (P) denote $\max\{c^\top x : Ax \le b\}$.

Cutting Plane Scheme

$$\max \left\{c^\top x : Ax \le b, x \text{ integer}\right\} \qquad \text{(IP)}$$



- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.

Cutting Plane Scheme

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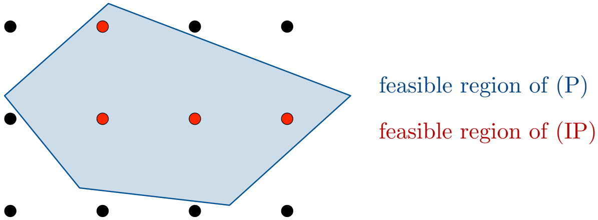

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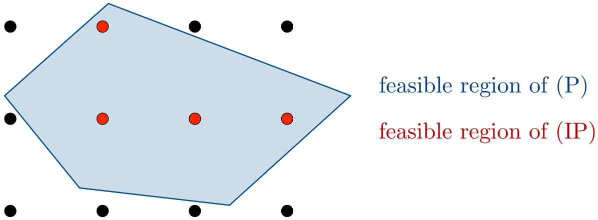

- Let (P) denote $\max\{c^\top x : Ax \leq b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.
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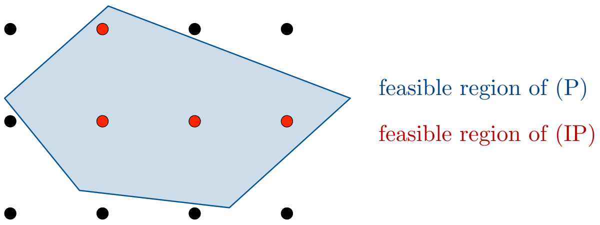
- If $\bar{x}$ is integral, then STOP. $\bar{x}$ is also optimal for (IP).

Cutting Plane Scheme

$$\max \left\{ c^\top x : Ax \le b, x \text{ integer} \right\} \qquad \text{(IP)}$$



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- Find a cutting plane $a^\top x \le \beta$ for $\bar{x}$.

Cutting Plane Scheme

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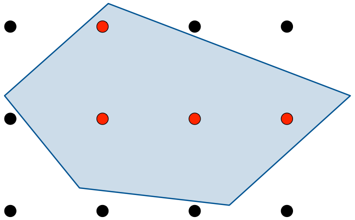

feasible region of (P)

feasible region of (IP)

- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.
- Let $\bar{x}$ be the optimal solution to (P).
- If $\bar{x}$ is integral, then STOP. $\bar{x}$ is also optimal for (IP).
- Find a cutting plane $a^\top x \le \beta$ for $\bar{x}$.
- Add constraint $a^\top x \le \beta$ to the system $Ax \le b$.

Question

How can we find cutting planes?

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How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Question

How can we find cutting planes?

Simplex does this for us!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

Question

How can we find cutting planes?

Simplex does this for us!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\lfloor 3.7 \rfloor \quad = \quad 3$$

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\begin{aligned} \lfloor 3.7 \rfloor &= 3 \\ \lfloor 62 \rfloor &= 62 \end{aligned}$$

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\begin{aligned} \lfloor 3.7 \rfloor &= 3 \\ \lfloor 62 \rfloor &= 62 \\ \lfloor -2.1 \rfloor &= -3 \end{aligned}$$

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$x \geq \mathbf{0}, \ x \text{ integer}$$

$$\boxed{\begin{array}{l} \max \; \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{array}{c} (1) \\ (2) \end{array} \\ \qquad x \ge \mathbf{0},\; x \text{ integer} \end{array}}$$

Add a slack variable, $x_3 \ge 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

$$
\begin{aligned}
&\max \ \begin{pmatrix}2 & 5\end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix}1 & 4 \\ 1 & 1\end{pmatrix} x \leq \begin{pmatrix}8 \\ 4\end{pmatrix} \qquad \begin{matrix}(1) \\ (2)\end{matrix} \\
&\qquad x \geq \mathbf{0},\ x \text{ integer}
\end{aligned}
$$

Add a slack variable, $x_3 \geq 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \geq 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

$$
\begin{aligned}
&\max \ \begin{pmatrix}2 & 5\end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix}1 & 4 \\ 1 & 1\end{pmatrix} x \le \begin{pmatrix}8 \\ 4\end{pmatrix} \qquad \begin{matrix}(1) \\ (2)\end{matrix} \\
&\qquad x \ge \mathbf{0}, \ x \text{ integer}
\end{aligned}
$$

Add a slack variable, $x_3 \ge 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \ge 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

Since x_1, x_2 are integers, $x_3 = 8 - x_1 - 4x_2$ and $x_4 = 4 - x_1 - x_2$ are integers.

$$
\begin{aligned}
&\max\ \begin{pmatrix}2 & 5\end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix} x \le \begin{pmatrix}8\\ 4\end{pmatrix} \qquad \begin{matrix}(1)\\ (2)\end{matrix} \\
&\qquad x \ge \mathbf{0},\ x \text{ integer}
\end{aligned}
$$

Add a slack variable, $x_3 \ge 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \ge 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

Since x_1, x_2 are integers, $x_3 = 8 - x_1 - 4x_2$ and $x_4 = 4 - x_1 - x_2$ are integers.

Thus, we can rewrite the IP as

$$
\begin{aligned}
&\max\ \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\ 4\end{pmatrix} \\
&\qquad x \ge \mathbf{0},\ x \text{ integer}
\end{aligned}
$$

Solving the IP

$$\max \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x$$

s. t.

$$\begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\ 4\end{pmatrix}$$

$$x \geq \mathbf{0},\ x \text{ integer}$$

Solving the IP

$$
\begin{aligned}
\max \ & \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\
\text{s. t.} \ & \\
& \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\
& x \geq \mathbf{0}, \ x \text{ integer}
\end{aligned}
$$

We will now relax the integer program.

Solving the IP

$$\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Solving the IP

$$\max \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x$$

s. t.

$$\begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\ 4\end{pmatrix}$$

$$x \geq \mathbf{0}$$

We will use the Simplex algorithm to solve this.

Solving the IP

$$\max \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x$$

s. t.

$$\begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\ 4\end{pmatrix}$$

$$x \geq \mathbf{0}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

Solving the IP

$$\begin{aligned} \max\ & \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.}\ & \\ & \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{aligned} \max\ & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.}\ & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

Solving the IP

$$
\begin{aligned}
&\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\
&\qquad x \geq \mathbf{0}
\end{aligned}
$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$
\begin{aligned}
&\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
&\qquad x \geq \mathbf{0}
\end{aligned}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Solving the IP

$$\begin{array}{l} \max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$. NOT INTEGER

Solving the IP

$$\begin{array}{l} \max \ \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{array}{l} \max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$. Not integer

Let us use the canonical form to get a cutting plane for $\bar{x}$.

$$
\begin{aligned}
\max\ & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.}\ & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

$$
\begin{aligned}
\max \; & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \; & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{8}{3}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$\begin{array}{l} \max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \quad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \quad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

$$\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

For every feasible solution to the IP, $x_1 - x_3 + x_4$ is integer.

$$
\boxed{
\begin{array}{l}
\max \; \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$
x_1 - \frac{1}{3} x_3 + \frac{4}{3} x_4 \leq \frac{8}{3}
$$

$$
x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}
$$

$$
x_1 - x_3 + x_4 \leq \frac{8}{3}
$$

For every feasible solution to the IP, $x_1 - x_3 + x_4$ is integer.

Hence, every feasible solution to the IP satisfies

$$
x_1 - x_3 + x_4 \leq \left\lfloor \frac{8}{3} \right\rfloor = 2
$$

$$
\begin{array}{l}
\max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

$$
\begin{array}{|l|}
\hline
\max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0} \\
\hline
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\boxed{\begin{array}{l} \max \begin{pmatrix}0 & 0 & -1 & -1\end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \qquad (\star)$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

$\Rightarrow$ $(\star)$ is a cutting plane for $\bar{x}$.

$$\boxed{\begin{array}{l} \max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

$\Rightarrow$ $(\star)$ is a cutting plane for $\bar{x}$.

We can rewrite $(\star)$ as

$$x_1 - x_3 + x_4 + x_5 = 2 \qquad \text{where} \qquad x_5 \geq 0.$$

$$
\boxed{\begin{array}{l}
\max \; \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$
x_1 - x_3 + x_4 \leq 2 \tag{$\star$}
$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$
\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2
$$

$\Rightarrow$ ($\star$) is a cutting plane for $\bar{x}$.

We can rewrite ($\star$) as

$$
x_1 - x_3 + x_4 + x_5 = 2 \qquad \text{where} \qquad x_5 \geq 0.
$$

We now add this to the relaxation.

$$
\begin{aligned}
\max \quad & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} \quad & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}
$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Solve this using the Simplex algorithm.

$$\max \quad \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12$$

s. t.

$$\begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$\begin{array}{ll}\max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\ & x \geq \mathbf{0}\end{array}$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$\begin{array}{ll}\max & \begin{pmatrix}0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2}\end{pmatrix} x + 11 \\ \text{s. t.} & \\ & \begin{pmatrix}1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2\end{pmatrix} x = \begin{pmatrix}3 \\ 1 \\ 1\end{pmatrix} \\ & x \geq \mathbf{0}\end{array}$$

$$
\begin{array}{ll}
\max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{ll}
\max & \begin{pmatrix}0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2}\end{pmatrix} x + 11 \\
\text{s. t.} & \\
& \begin{pmatrix}1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2\end{pmatrix} x = \begin{pmatrix}3 \\ 1 \\ 1\end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$.

$$
\begin{array}{ll}
\max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{ll}
\max & \begin{pmatrix}0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2}\end{pmatrix} x + 11 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$. INTEGER!

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} x + 11 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

$(3,1,1,0,0)^\top$ is optimal for

$$\begin{aligned}
\max\ & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\
\text{s. t.}\ & \\
& \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\
& x \geq \mathbf{0}, \quad x \text{ integer}
\end{aligned}$$

$(3,1,1,0,0)^\top$ is optimal for

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}, \quad x \text{ integer}
\end{array}
$$

$(3,1)^\top$ is optimal for

$$
\begin{array}{ll}
\max & \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
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\end{array}
$$

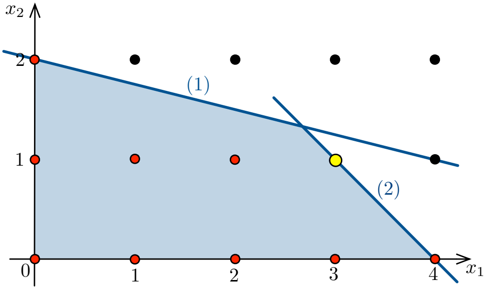
$(3,1,1,0,0)^\top$ is optimal for

$$\begin{aligned}&\max\ \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12\\ &\text{s. t.}\\ &\qquad \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0\\ 0 & 1 & 1/3 & -1/3 & 0\\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3\\ 4/3\\ 2\end{pmatrix}\\ &\qquad x \geq \mathbf{0},\quad x \text{ integer}\end{aligned}$$

$\Rightarrow$

$(3,1)^\top$ is optimal for

$$\begin{aligned}&\max\ \begin{pmatrix}2 & 5\end{pmatrix} x\\ &\text{s. t.}\\ &\qquad \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix} x = \begin{pmatrix}8\\ 4\end{pmatrix} \qquad \begin{matrix}(1)\\ (2)\end{matrix}\\ &\qquad x \geq \mathbf{0},\quad x \text{ integer}\end{aligned}$$



Getting Cutting Planes in General

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Solve the relaxation and get the LP in a canonical form for B.

$$
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$\Rightarrow$ $(\star)$ is a cutting plane for $\bar{x}$.

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Ways we can Improve the Algorithm

- Do not use the 2-phase Simplex to reoptimize; work with the dual.
- Add more than one cutting plane at at time.
- Combine it with a divide and conquer strategy (branch and bound).

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- Cutting planes can be obtained from the final canonical form.
- Careful implementation is key to success.

Module 6: Nonlinear Programs (Convexity)

Definition

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This is a very general model, but NLPs can be very hard to solve!

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$$
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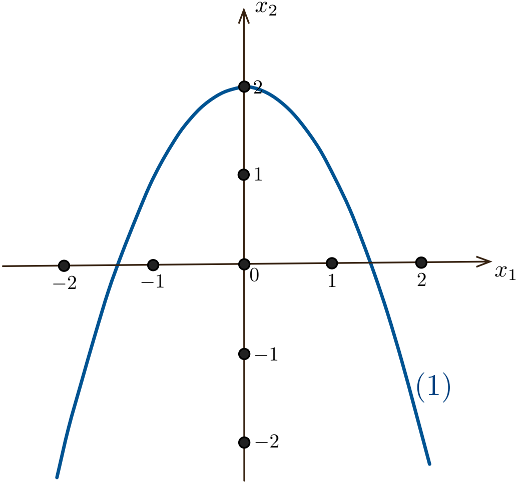
$$



(1) $x_2 \geq 2 - x_1^2$.

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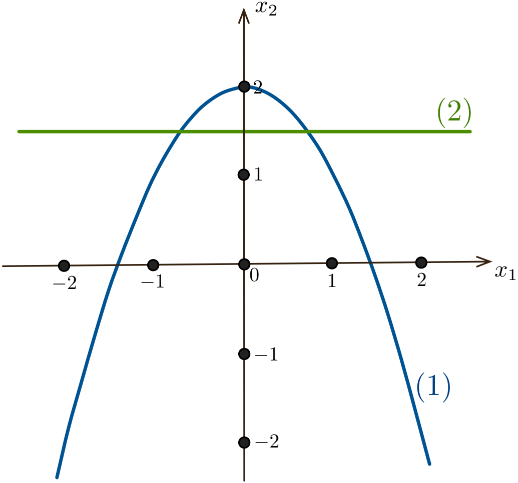

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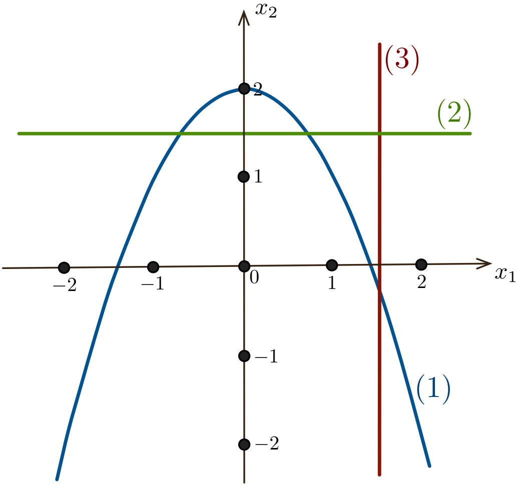

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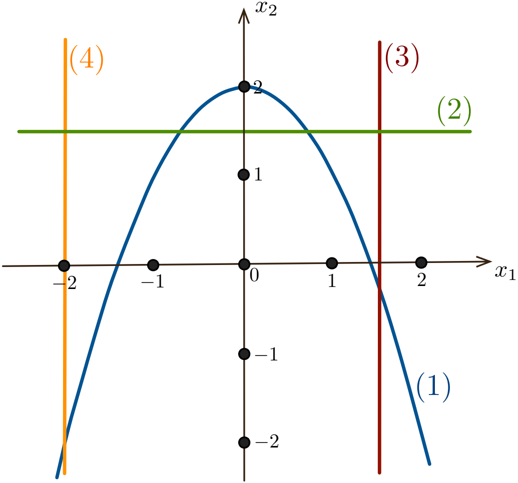

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$$
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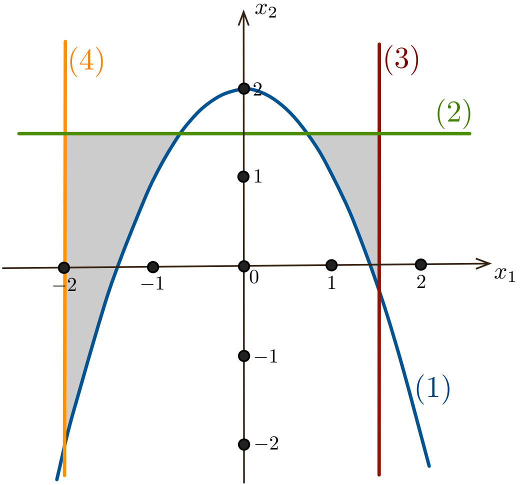

Feasible region

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$$\boxed{\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \le 0 & (i = 1, \ldots, k) \end{array}} \tag{P}$$

Remark

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The optimal solution to (Q) will have $\lambda = f(x)$.

Nonlinear Programs Generalize Linear Programs

$$\max \quad x_1 + x_2$$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &= 4 \end{aligned}$$

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Nonlinear Programs can also generalize Integer Programs!

Nonlinear Programs Generalize **Integer** Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \in \{0,1\} \quad (j = 1, \ldots, n) \end{array}$$

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Quadratic NLP

Remark

$0, 1$ IPs are hard to solve; thus, quadratic NLPs are also hard to solve.

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Remark

IPs are hard to solve; thus, NLPs are also hard to solve.

Question

What makes solving an NLP hard?

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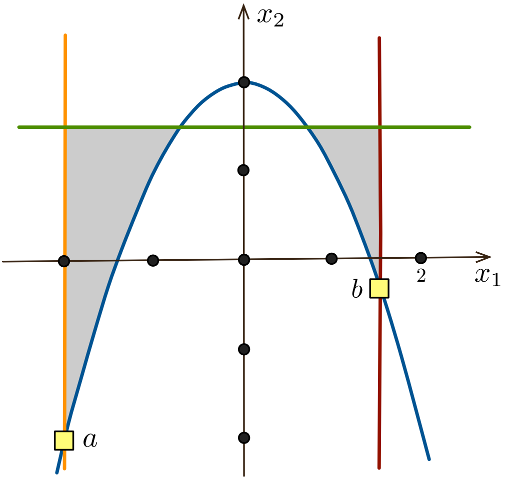

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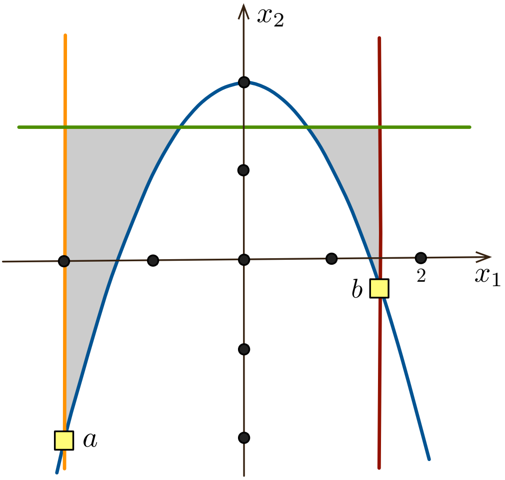

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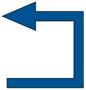

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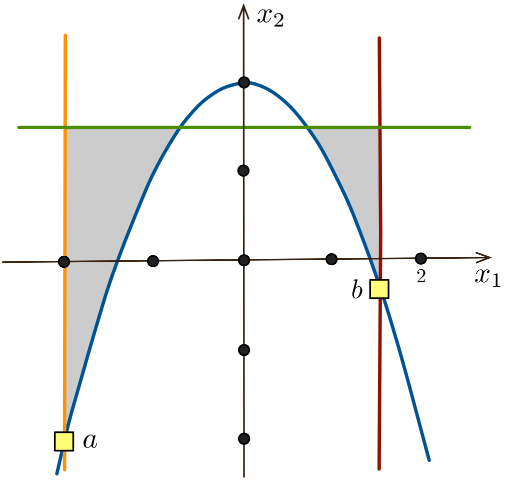

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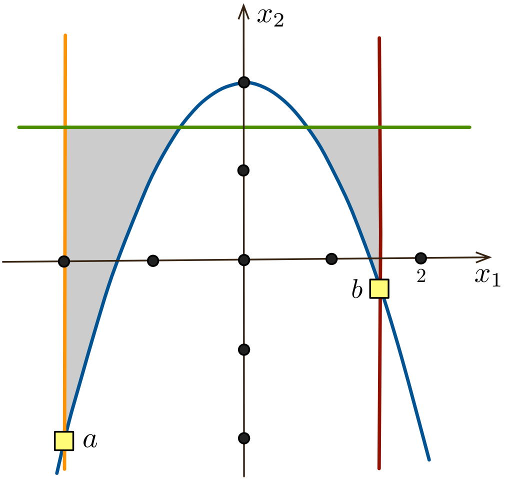

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$$\min\left\{f(x) : x \in S\right\}. \tag{P}$$

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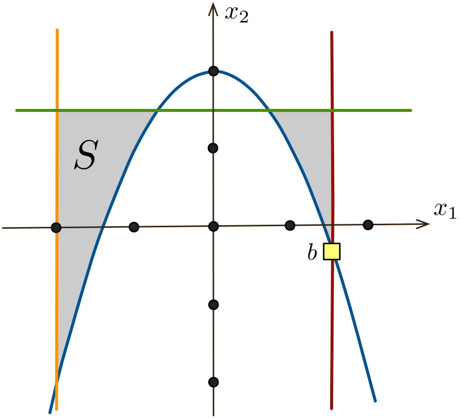

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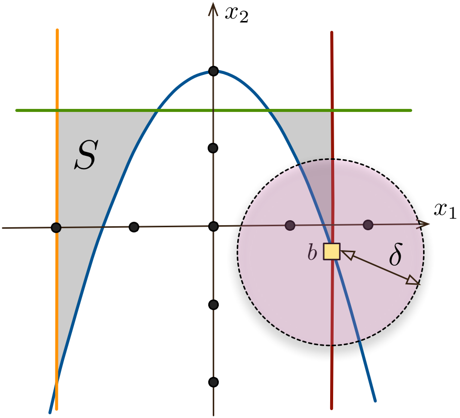

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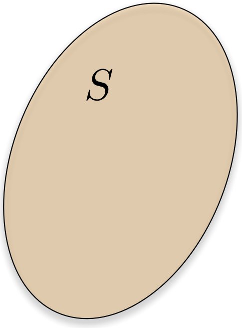

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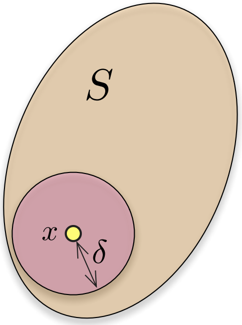

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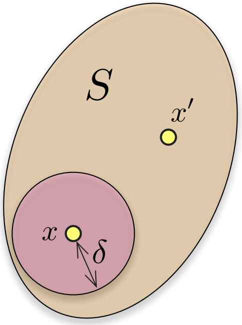

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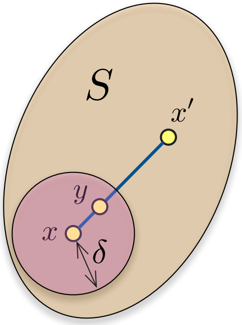

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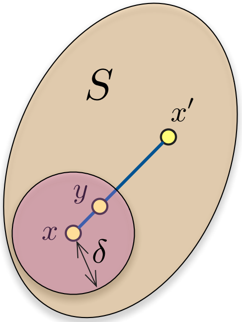

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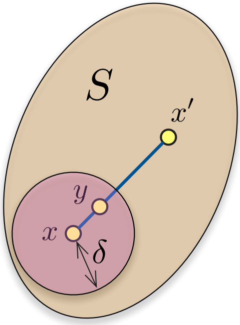

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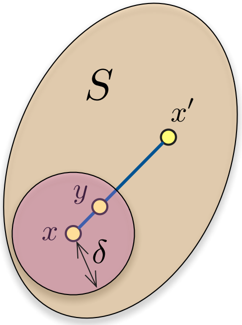
$$c^\top y$$



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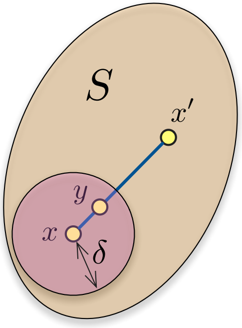

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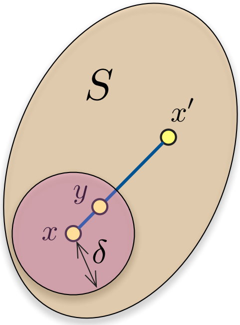

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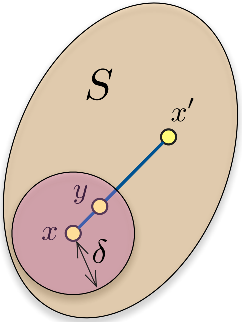

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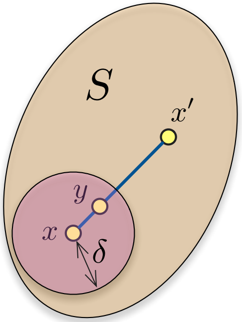

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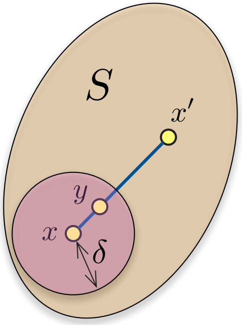
A contradiction.



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Goal: Study a case where the feasible region of (P) is convex.

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- We will define convex functions
- We will prove

Proposition

If $g_1, \ldots, g_k$ are all convex, then the feasible region of (P) is convex.

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Function $f : \Re^n \to \Re$ is convex

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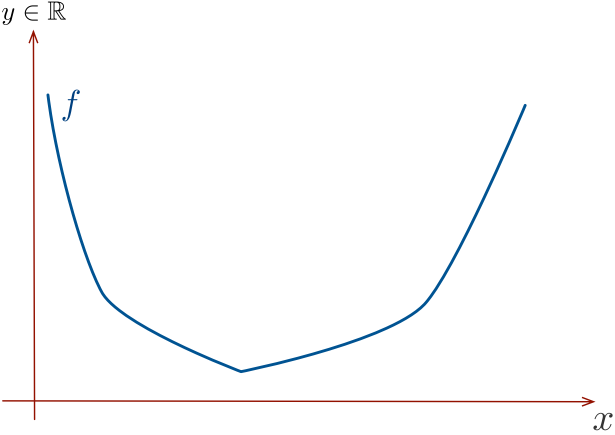

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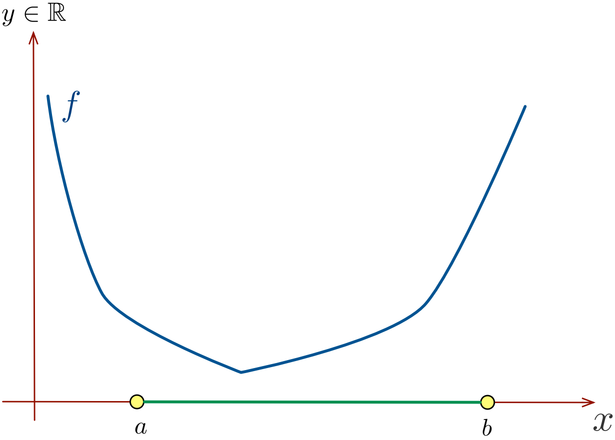

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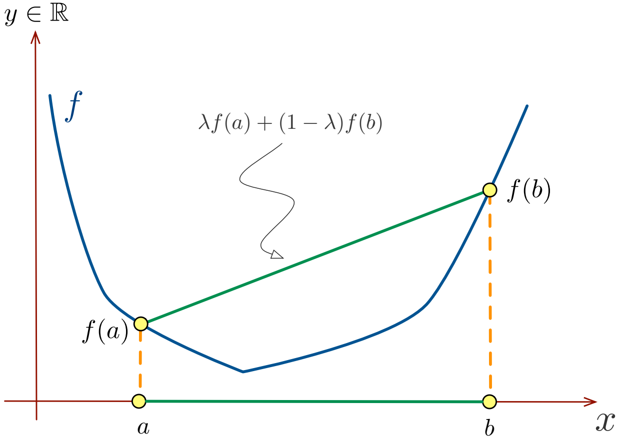

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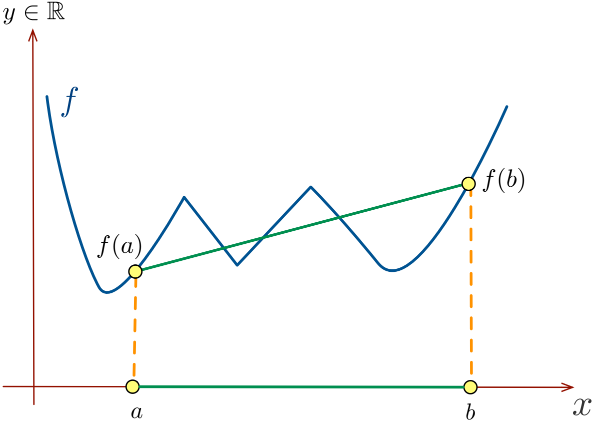

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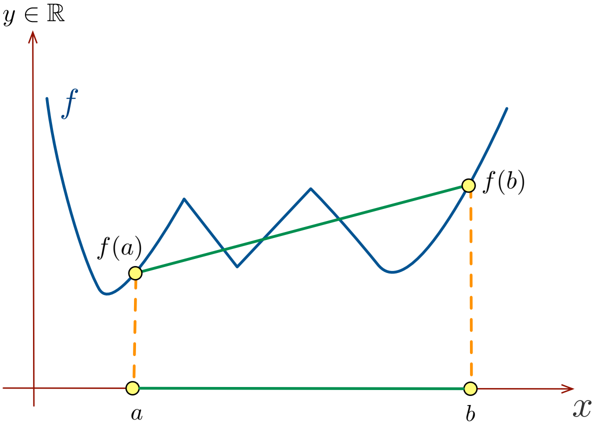

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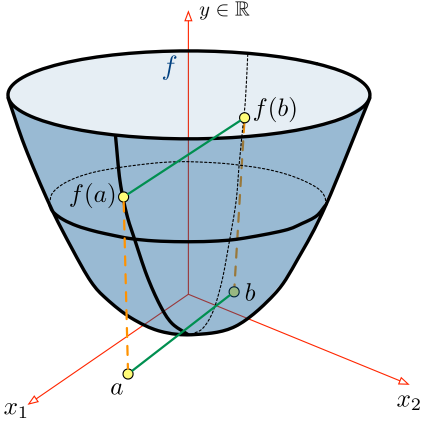

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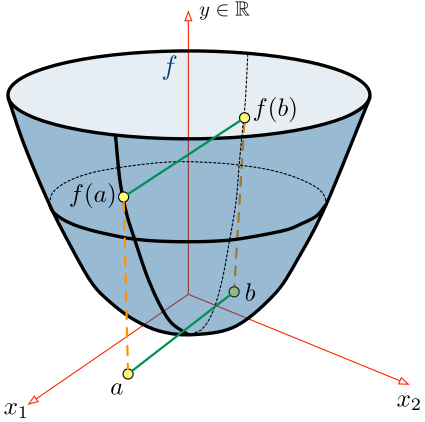

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which is the case as $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$.

Why Do We Care About Convex Functions?

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Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

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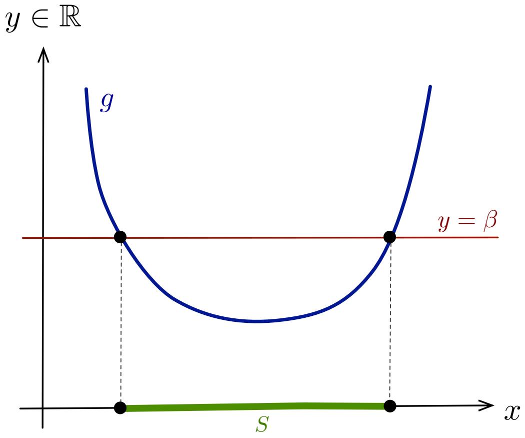

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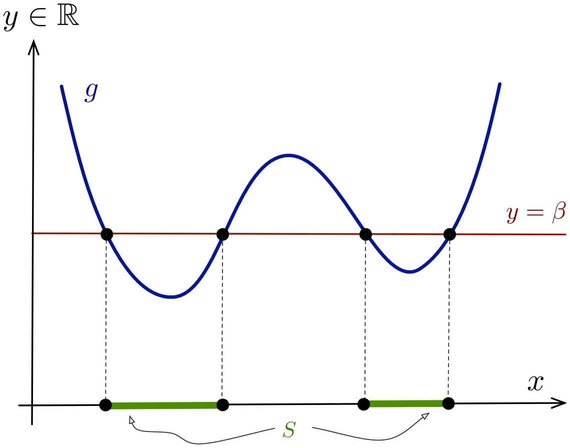

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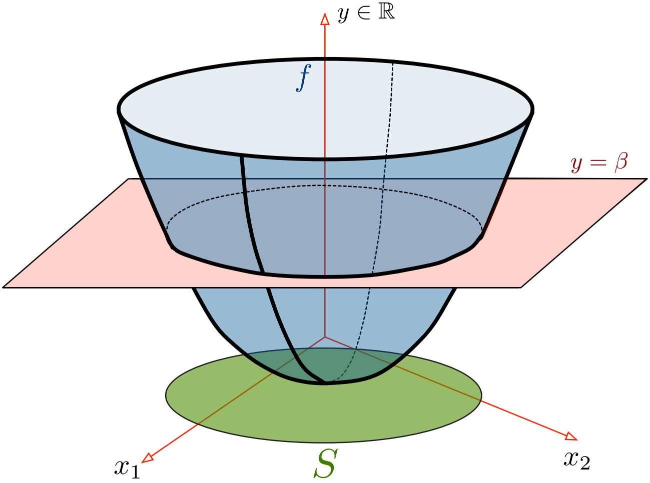

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Since the intersection of convex sets is convex, the result follows.

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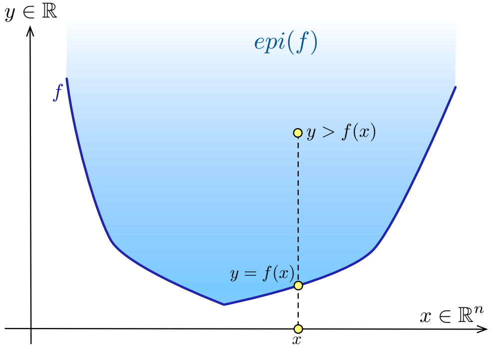

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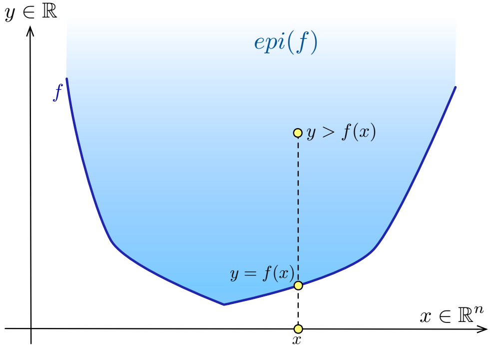

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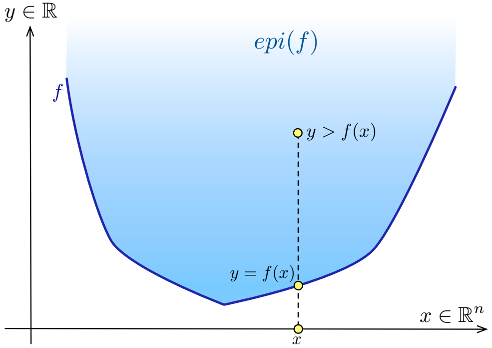

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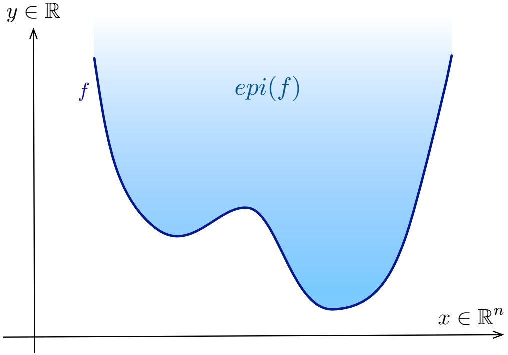

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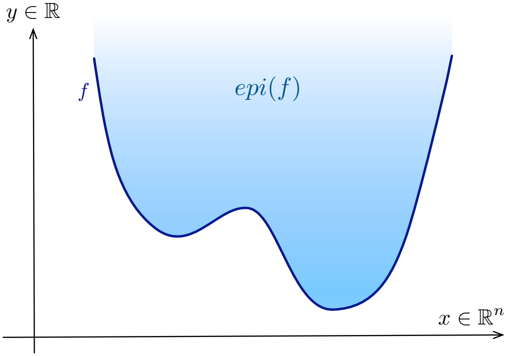

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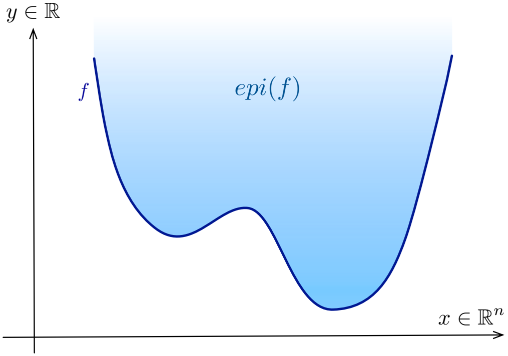

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Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \leq \lambda \leq 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix} \alpha \\ a \end{pmatrix} + (1-\lambda) \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} \lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b \end{pmatrix} \qquad (\star)$$

Consider

$$f\left(\lambda a + (1-\lambda)b\right) \quad \leq \quad \text{(convexity of } f\text{)}$$
$$\lambda f(a) + (1-\lambda) f(b)$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad f$ is convex.

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Consider

$$f\left(\lambda a + (1-\lambda) b\right) \quad \leq \quad \text{(convexity of } f\text{)}$$

$$\underbrace{\lambda}_{\geq 0} f(a) + \underbrace{(1-\lambda)}_{\geq 0} f(b)$$

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Consider

$$\begin{aligned} & f\left(\lambda a + (1-\lambda) b\right) && \leq && \text{(convexity of } f\text{)} \\ & \underbrace{\lambda}_{\geq 0} \underbrace{f(a)}_{\leq \alpha} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{f(b)}_{\leq \beta} && \leq && \\ & \lambda\alpha + (1-\lambda)\beta. \end{aligned}$$

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Consider

$$\begin{aligned} f\left(\lambda a + (1-\lambda) b\right) \quad &\le \quad \text{(convexity of } f\text{)} \\ \underbrace{\lambda}_{\ge 0} \underbrace{f(a)}_{\le \alpha} + \underbrace{(1-\lambda)}_{\ge 0} \underbrace{f(b)}_{\le \beta} \quad &\le \\ \lambda\alpha + (1-\lambda)\beta. \end{aligned}$$

Thus $(\star)$ is in $epi(f)$.

Recap

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1. NLPs are hard in general.

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3. Local optimum = optimal sol when the feasible region is convex.
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5. Convex functions yield a convex feasible region.
6. Convex functions and convex sets are related by epigraphs.

Module 6: Nonlinear Programs (the KKT theorem)

$$
\begin{array}{llll}
\min & -x_1 - x_2 & & \\
\text{s.t.} & & & \\
& -x_2 + x_1^2 & \leq 0 & (1) \\
& -x_1 + x_2^2 & \leq 0 & (2) \\
& -x_1 + \frac{1}{2} & \leq 0 & (3)
\end{array}
$$

$$
\begin{array}{lrcll}
\min & -x_1 - x_2 & & & \\
\text{s.t.} & & & & \\
& -x_2 + x_1^2 & \leq & 0 & (1) \\
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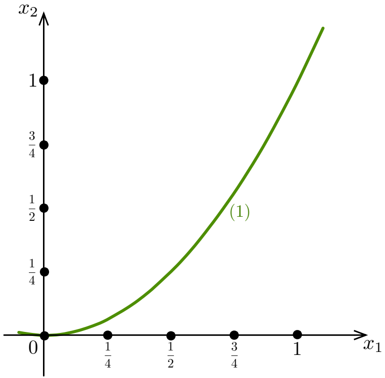
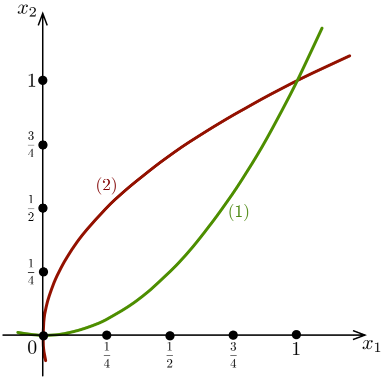
& -x_1 + \frac{1}{2} & \leq & 0 & (3)
\end{array}
$$



(1) $x_2 \geq x_1^2$;

$$
\begin{array}{llll}
\min & -x_1 - x_2 & & \\
\text{s.t.} & & & \\
& -x_2 + x_1^2 & \leq & 0 \quad (1) \\
& -x_1 + x_2^2 & \leq & 0 \quad (2) \\
& -x_1 + \frac{1}{2} & \leq & 0 \quad (3)
\end{array}
$$



(1) $x_2 \geq x_1^2$;

(2) $x_1 \geq x_2^2$;

$$
\begin{array}{lrcll}
\min & -x_1 - x_2 & & & \\
\text{s.t.} & & & & \\
& -x_2 + x_1^2 & \leq & 0 & (1) \\
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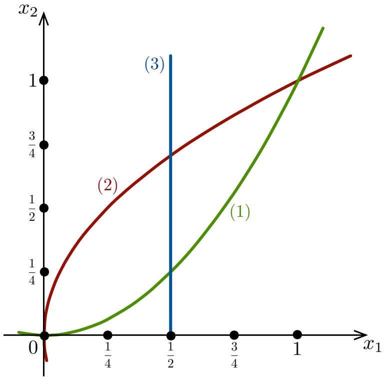
& -x_1 + \frac{1}{2} & \leq & 0 & (3)
\end{array}
$$

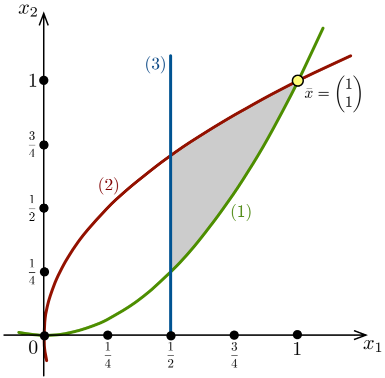


(1) $x_2 \geq x_1^2$;

(2) $x_1 \geq x_2^2$;

(3) $x_1 \geq \frac{1}{2}$.

$$
\begin{array}{lrcll}
\min & -x_1 - x_2 & & & \\
\text{s.t.} & & & & \\
& -x_2 + x_1^2 & \leq & 0 & (1) \\
& -x_1 + x_2^2 & \leq & 0 & (2) \\
& -x_1 + \frac{1}{2} & \leq & 0 & (3)
\end{array}
$$



$$
\begin{array}{lrcll}
\min & -x_1 - x_2 & & & \\
\text{s.t.} & & & & \\
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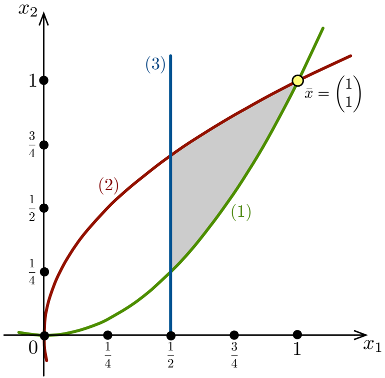
$$



Claim

$\bar{x} = (1, 1)^\top$ is an optimal solution to the NLP.

$$
\begin{array}{llll}
\min & -x_1 - x_2 & & \\
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& -x_2 + x_1^2 & \leq \ 0 & (1) \\
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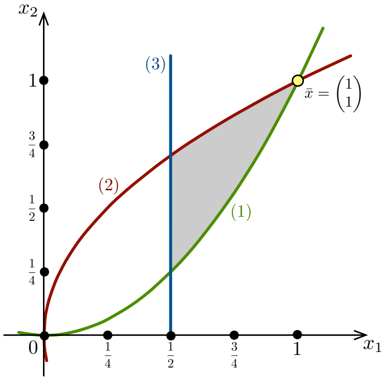
$$



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How do we prove this?

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\begin{array}{llcl}
\min & -x_1 - x_2 & & \\
\text{s.t.} & & & \\
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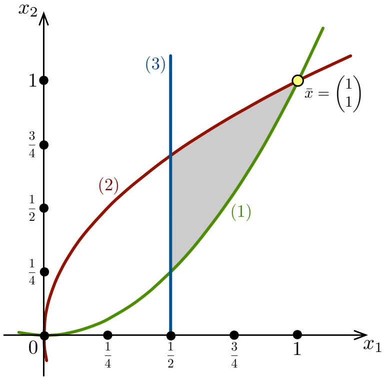
\end{array}
$$



Claim

$\bar{x} = (1, 1)^\top$ is an optimal solution to the NLP.

How do we prove this?

Step 1. Find a relaxation of the NLP.

$$
\begin{array}{llll}
\min & -x_1 - x_2 & & \\
\text{s.t.} & & & \\
& -x_2 + x_1^2 & \leq 0 & (1) \\
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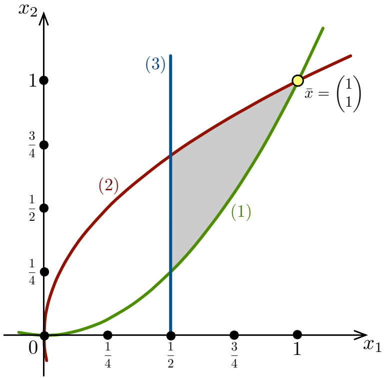
& -x_1 + \frac{1}{2} & \leq 0 & (3)
\end{array}
$$



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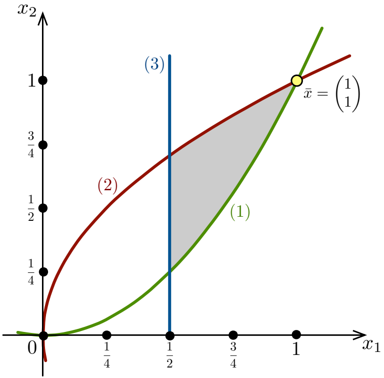
How do we prove this?

Step 1. Find a relaxation of the NLP.

Step 2. Prove $\bar{x}$ is optimal for the relaxation.

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & -x_2 + x_1^2 & \leq & 0 & (1) \\ & -x_1 + x_2^2 & \leq & 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq & 0 & (3) \end{array}$$



Claim

$\bar{x} = (1,1)^\top$ is an optimal solution to the NLP.

How do we prove this?

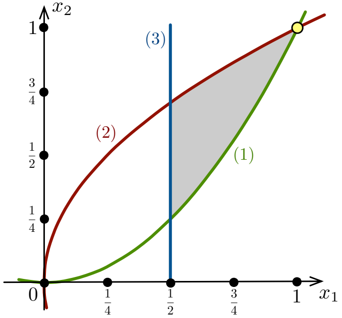
Step 1. Find a relaxation of the NLP.

Step 2. Prove $\bar{x}$ is optimal for the relaxation.

Step 3. Deduce that $\bar{x}$ is optimal for the NLP.

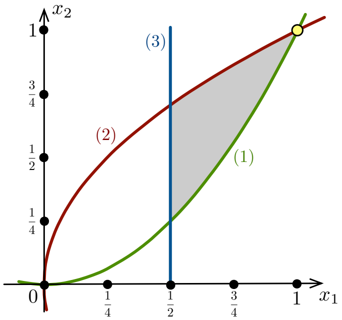
Original NLP

$$
\begin{array}{llcll}
\min & -x_1 - x_2 & & & \\
\text{s.t.} & & & & \\
& -x_2 + x_1^2 & \leq & 0 & (1) \\
& -x_1 + x_2^2 & \leq & 0 & (2) \\
& -x_1 + \frac{1}{2} & \leq & 0 & (3)
\end{array}
$$



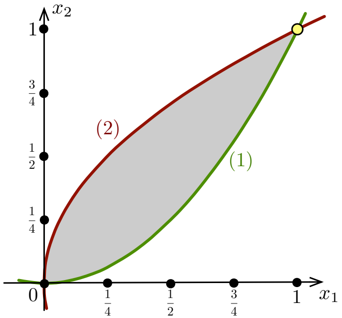
Original NLP

$$\begin{array}{llll} \min & -x_1 - x_2 & & \\ \text{s.t.} & & & \\ & -x_2 + x_1^2 & \leq 0 & (1) \\ & -x_1 + x_2^2 & \leq 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq 0 & (3) \end{array}$$



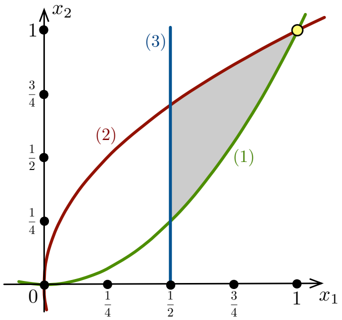
Relaxation

$$\begin{array}{llll} \min & -x_1 - x_2 & & \\ \text{s.t.} & & & \\ & -x_2 + x_1^2 & \leq 0 & (1) \\ & -x_1 + x_2^2 & \leq 0 & (2) \end{array}$$



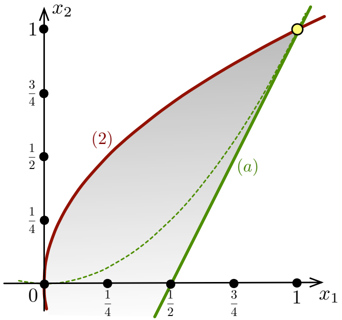
Original NLP

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & -x_2 + x_1^2 & \leq & 0 & (1) \\ & -x_1 + x_2^2 & \leq & 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq & 0 & (3) \end{array}$$



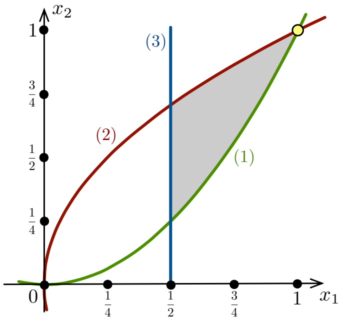
New relaxation

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & 2x_1 - x_2 & \leq & 1 & (a) \\ & -x_1 + x_2^2 & \leq & 0 & (2) \end{array}$$



Original NLP

$$\begin{array}{lllll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & -x_2 + x_1^2 & \leq & 0 & (1) \\ & -x_1 + x_2^2 & \leq & 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq & 0 & (3) \end{array}$$



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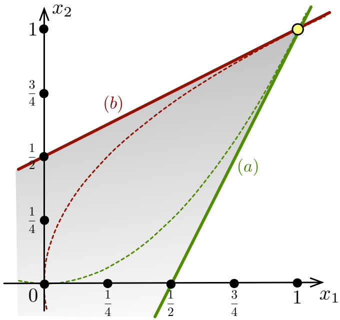

Claim

$\bar{x} = (1,1)^\top$ is an optimal solution to

$$\boxed{\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & 2x_1 - x_2 & \leq & 1 & (a) \\ & -x_1 + 2x_2 & \leq & 1 & (b) \end{array}}$$

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Proof

Claim

$\bar{x} = (1,1)^\top$ is an optimal solution to

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Proof

Tight constraints for $\bar{x}$ are (a) and (b).

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Goal: Show that the objective function is in the cone of tight constraints.

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$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \overset{?}{\in} cone \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

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$\bar{x} = (1,1)^\top$ is an optimal solution to

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$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \times \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 1 \times \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \checkmark$$

Original NLP

$$
\begin{array}{lrcll}
\min & -x_1 - x_2 & & & \\
\text{s.t.} & & & & \\
& -x_2 + x_1^2 & \leq & 0 & (1) \\
& -x_1 + x_2^2 & \leq & 0 & (2) \\
& -x_1 + \frac{1}{2} & \leq & 0 & (3)
\end{array}
$$

Relaxation

$$
\begin{array}{lrcll}
\min & -x_1 - x_2 & & & \\
\text{s.t.} & & & & \\
& 2x_1 - x_2 & \leq & 1 & (a) \\
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$$

Original NLP

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Relaxation

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & 2x_1 - x_2 & \leq & 1 & (a) \\ & -x_1 + 2x_2 & \leq & 1 & (b) \end{array}$$

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Original NLP

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Relaxation

$$\begin{array}{llrcll} \min & -x_1 - x_2 & & & & \\ \text{s.t.} & & & & & \\ & & 2x_1 - x_2 & \leq & 1 & (a) \\ & & -x_1 + 2x_2 & \leq & 1 & (b) \end{array}$$

$\bar{x} = (1,1)^\top$ is an optimal solution to the relaxation

$\Rightarrow$ $\bar{x}$ is an optimal solution to the *original NLP*

Original NLP

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & -x_2 + x_1^2 & \leq & 0 & (1) \\ & -x_1 + x_2^2 & \leq & 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq & 0 & (3) \end{array}$$

Relaxation

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & 2x_1 - x_2 & \leq & 1 & (a) \\ & -x_1 + 2x_2 & \leq & 1 & (b) \end{array}$$

$\bar{x} = (1,1)^\top$ is an optimal solution to the relaxation

$\Rightarrow$ $\bar{x}$ is an optimal solution to the *original NLP*

Question

Can we do this in general?

Original NLP

$$\begin{array}{llll} \min & -x_1 - x_2 & & \\ \text{s.t.} & & & \\ & -x_2 + x_1^2 & \leq 0 & (1) \\ & -x_1 + x_2^2 & \leq 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq 0 & (3) \end{array}$$

Relaxation

$$\begin{array}{llll} \min & -x_1 - x_2 & & \\ \text{s.t.} & & & \\ & 2x_1 - x_2 & \leq 1 & (a) \\ & -x_1 + 2x_2 & \leq 1 & (b) \end{array}$$

$\bar{x} = (1,1)^\top$ is an optimal solution to the relaxation

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Question

Can we do this in general? YES

Original NLP

$$\begin{array}{llrcll} \min & -x_1 - x_2 & & & & \\ \text{s.t.} & & & & & \\ & & -x_2 + x_1^2 & \leq & 0 & (1) \\ & & -x_1 + x_2^2 & \leq & 0 & (2) \\ & & -x_1 + \frac{1}{2} & \leq & 0 & (3) \end{array}$$

Relaxation

$$\begin{array}{llrcll} \min & -x_1 - x_2 & & & & \\ \text{s.t.} & & & & & \\ & & 2x_1 - x_2 & \leq & 1 & (a) \\ & & -x_1 + 2x_2 & \leq & 1 & (b) \end{array}$$

$\bar{x} = (1,1)^\top$ is an optimal solution to the relaxation

$\Rightarrow$ $\bar{x}$ is an optimal solution to the *original NLP*

Question

Can we do this in general? YES

The key tool we'll use is subgradients.

Definition

Let $f : \Re^n \to \Re$ be a convex function

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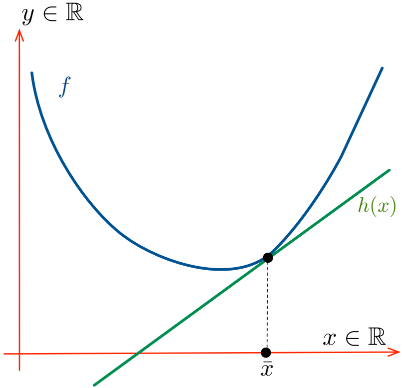

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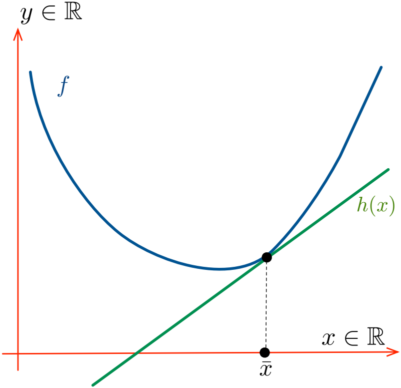
unique subgradient



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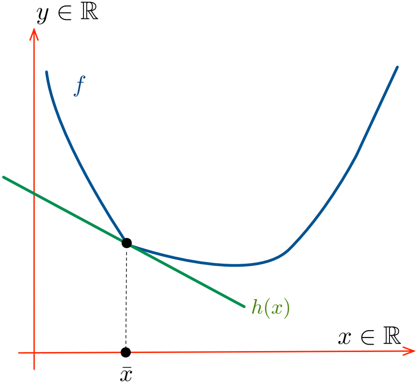

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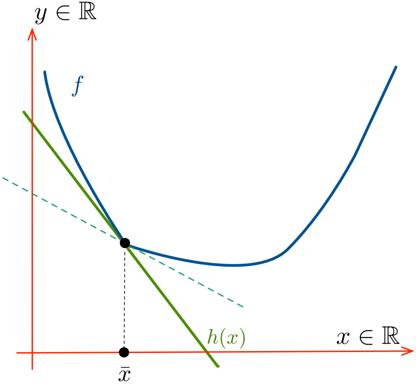

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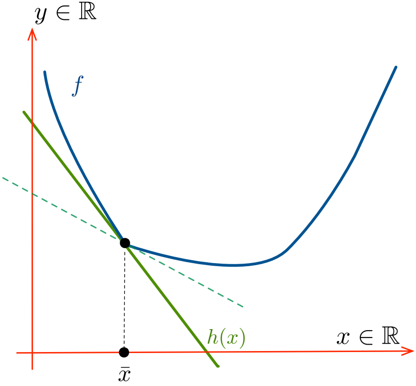

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which is the case as $x_2^2 - 2x_2 + 1 = (x_2 - 1)^2 \geq 0$.

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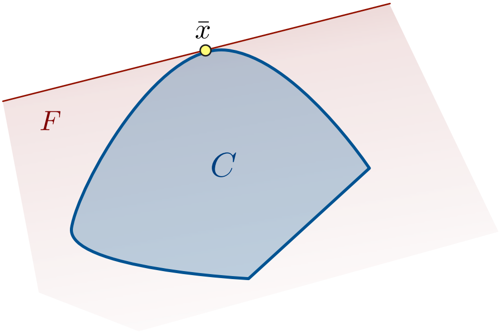

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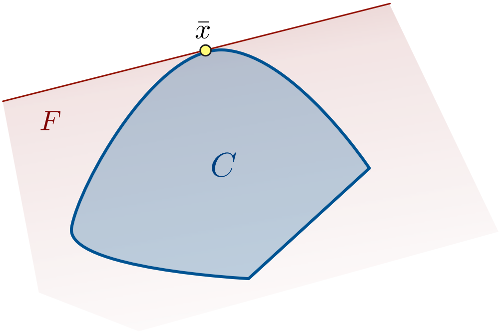

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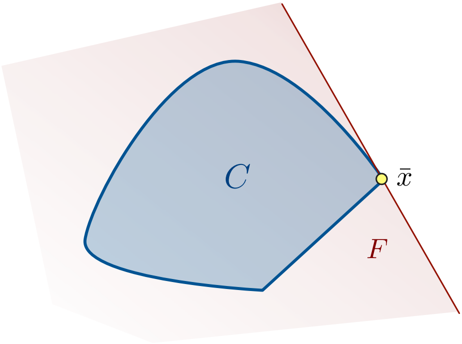

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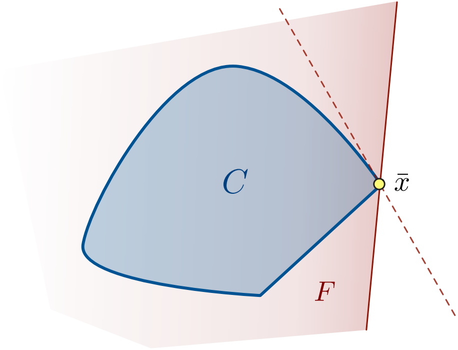

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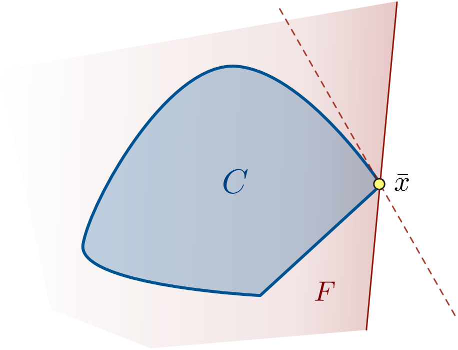

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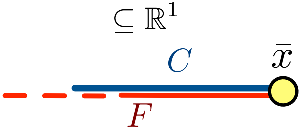

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- $h(\bar{x}) = g(\bar{x}) = 0$ thus, $\bar{x}$ is on the boundary of F.

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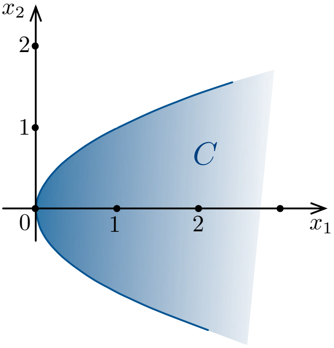

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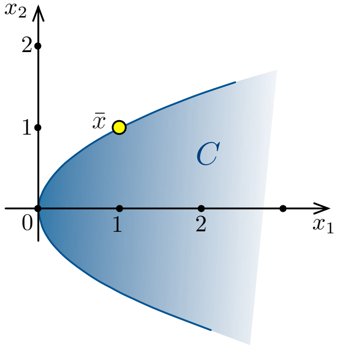

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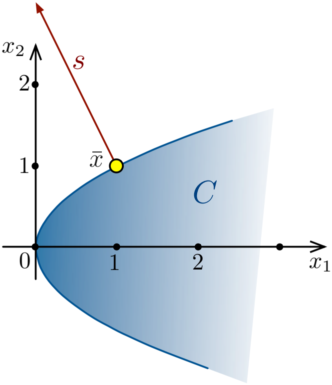

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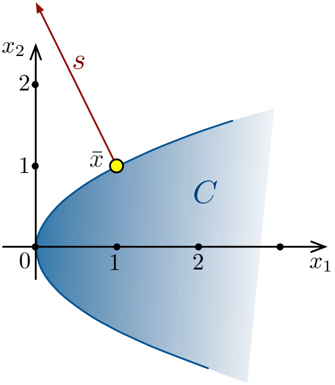

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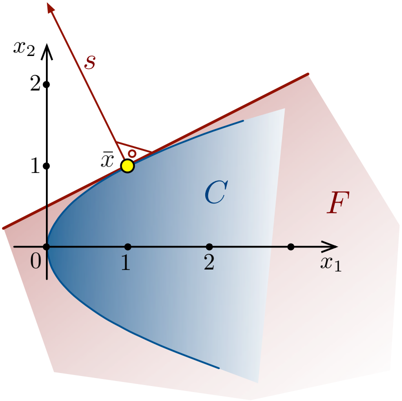

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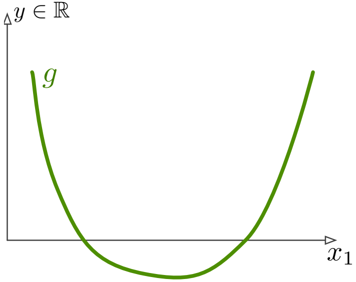

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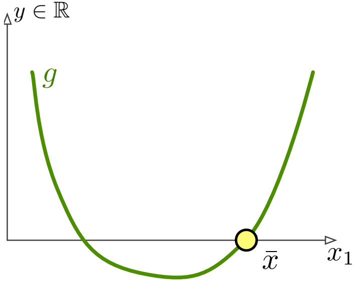

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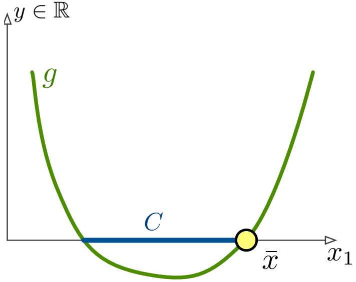

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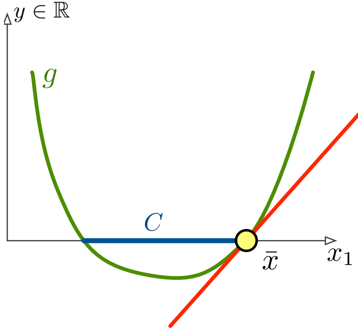

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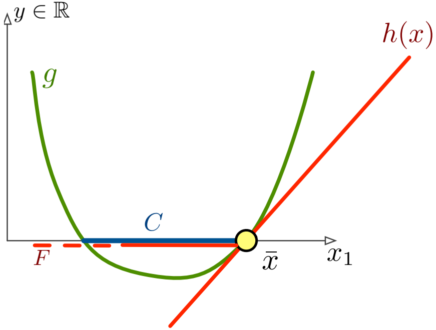

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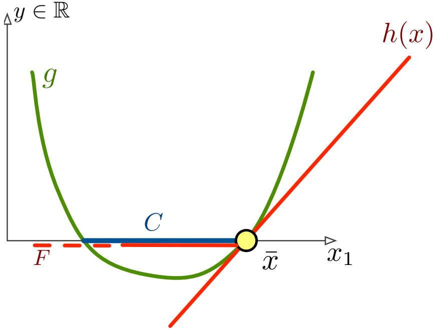

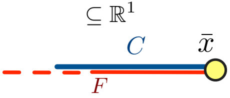

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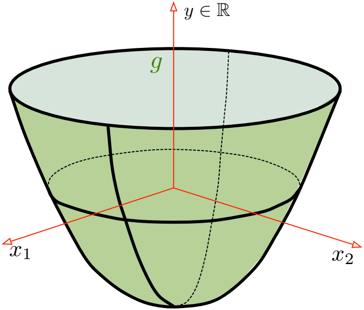

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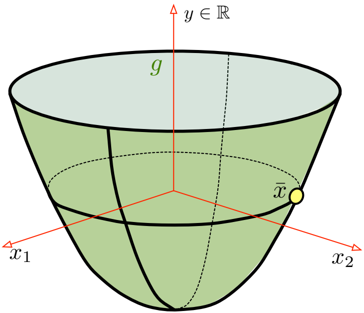

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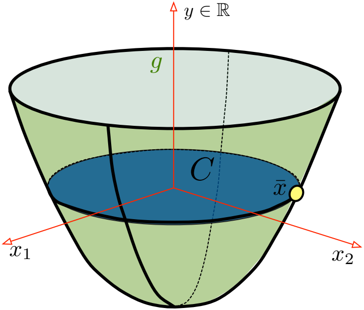

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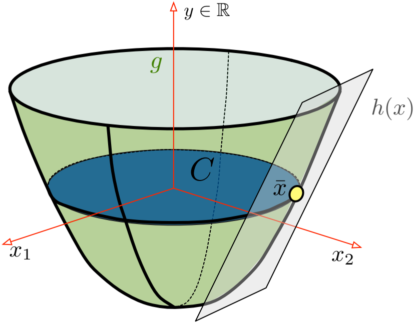

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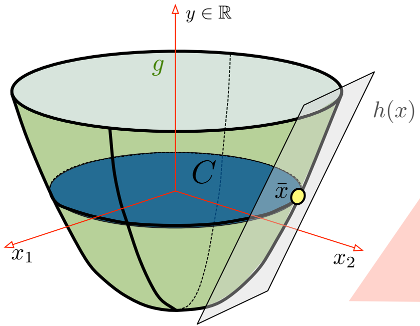

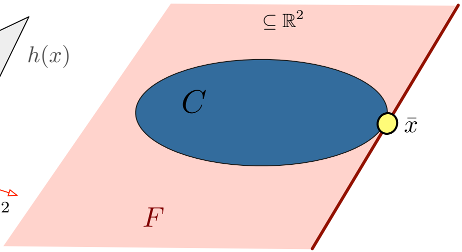

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Question

Why is this relevant for us?

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We use it to construct relaxations of NLPs

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If we replace the nonlinear constraint

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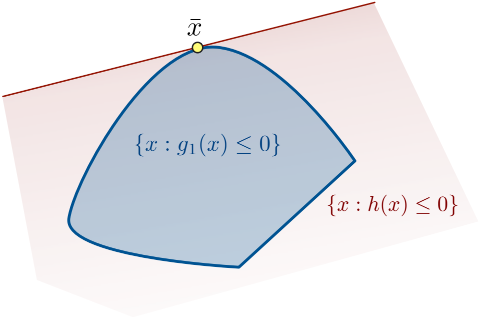

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If $-c \in cone\left\{s^{(i)} : i \in I\right\}$ then $\bar{x}$ is optimal.

Proposition

$$\min \quad c^\top x$$

s.t.

$$g_i(x) \le 0 \quad (i = 1, \dots, k)$$

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If $-c \in cone\left\{s^{(i)} : i \in I\right\}$ then $\bar{x}$ is optimal.

Example

$$\min \quad -x_1 - x_2$$

s.t.

$$\begin{aligned} -x_2 + x_1^2 &\le 0 \quad (1)\\ -x_1 + x_2^2 &\le 0 \quad (2)\\ -x_1 + \tfrac{1}{2} &\le 0 \quad (3) \end{aligned}$$

$\bar{x} = (1,1)^\top$ feasible

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$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \\ & g_i(x) \leq 0 \quad (i = 1, \ldots, k) \end{aligned}$$

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$$- \begin{pmatrix} -1 \\ -1 \end{pmatrix} \in cone \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} \qquad \Longrightarrow \qquad \bar{x} \text{ optimal.}$$

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We proved that the set of solutions to $g_i(x) \le 0$

is contained in the set of solutions to $g_i(\bar{x}) + s^{(i)}(x - \bar{x}) \le 0$.

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We have a relaxation

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This means that $\bar{x}$ is also optimal for the NLP.

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Question

Is there a converse to this result?

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Is there a converse to this result? YES

Gradients: A Calculus Detour

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Let $f : \Re^n \to \Re$ be a convex function and let $\bar{x} \in \Re^n$.

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If the gradient $\nabla f(\bar{x})$ of f exists at $\bar{x}$, then it is a subgradient.

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Proposition

Let $f : \Re^n \to \Re$ be function and let $\bar{x} \in \Re^n$.

If the partial derivative $\frac{\partial f(x)}{\partial x_j}$ exists for f at $\bar{x}$ for all $j = 1, \ldots, n$, then the gradient $\nabla f(\bar{x})$ is obtained by evaluating for $\bar{x}$,

$$\left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)^\top .$$

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Compute the gradient of the convex function

$$f(x) = -x_2 + x_1^2$$

at $\bar{x} = (1, 1)^\top$.

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Since $(2, -1)^\top$ is the gradient of f at $\bar{x}$, it is a subgradient as well.

Definition

A feasible solution to $\bar{x}$ is a Slater point of

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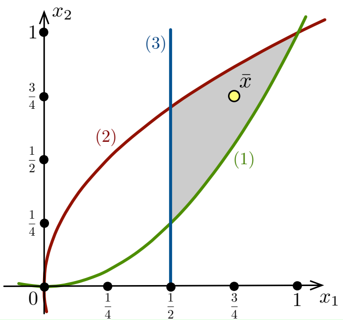

The Karush-Kuhn-Tucker (KKT) Theorem

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Remark

We proved the "easy" direction "$\Longleftarrow$".

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- We related subgradients and supporting halfspaces.
- We showed how to relax convex constraints by a linear constraint.
- We gave sufficient conditions for a solution to be optimal.
- We stated the KKT theorem.