# Introduction to Optimization

## Part 1: Formulations (Overview)

# Outline

# Optimization - Abstract Perspective

- Abstractly, we will focus on problems of the following form:
  - Given: set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$
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- Very general problem that is enormously useful in virtually very branch of industry.
- Bad news: the above problem is notoriously hard to solve (and may not even be well-defined).

# Optimization - Important Special Cases

- *Abstract optimization problem (P):*
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  (A) Linear Programming. $A$ is implicitly given by *linear* constraints, and $f$ is a *linear* function.

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  (B) Integer Programming. Same as before, but now we want to max/min over the *integer* points in $A$.
  (C) Nonlinear Programming. $A$ is given by *non-linear* constraints, and $f$ is a *non-linear* function.

# Optimization - Typical Workflow

Typical development process has three stages.

- Starting point is English language description of practical problem

**Practical Problem**:
- Description in plain language
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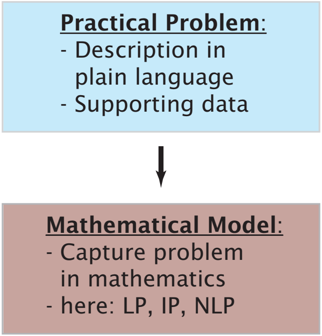

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↓

**Mathematical Model**:
- Capture problem in mathematics
- here: LP, IP, NLP

↓

**Practical Implementation**:
- E.g., OPL (IBM ILOG)

# Optimization - Typical Workflow

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- We will develop a mathematical model for the problem.
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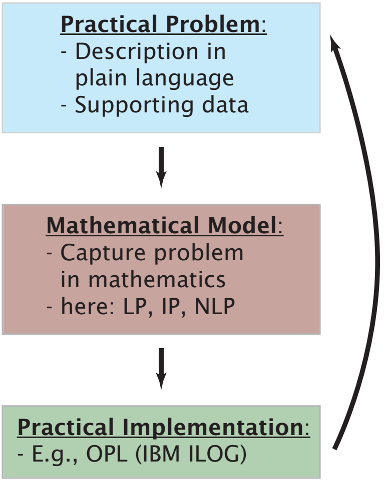
- Iterate!



# Optimization in Practice

Optimization is everywhere! Some examples:

- Booking hotel rooms or airline tickets,
- Setting the market price of a kwh of electricity,
- Determining an "optimal" portfolio of stocks,
- Computing energy efficient circuits in chip design,
- and many more!

# CSX Rail

- One of the largest transport suppliers in the United States.
- CSX operates 21000 miles of rail network
- 11 Billion in annual revenue
- Serves 23 states, Ontario and Quebec
- Operates 1200 trains per day


- Has a fleet of 3800 locomotives, and more than 100000 freight cars
- Transports 7.4 million car loads per year

# Optimization @ CSX Rail

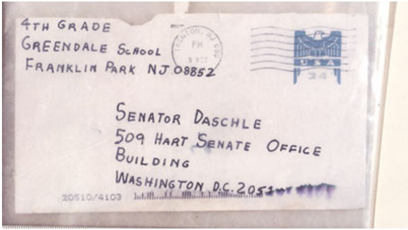
- [Acharya, Sellers, Gorman '10] use mathematical programming to optimally allocate and reposition empty railcars dynamically


  - Implementing system yields the following estimated benefits for CSX:

    Annual savings: $51 million
    Avoided rail car capital investment: $1.4 billion

# Optimization in Disease Control

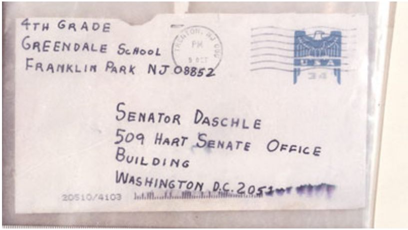
- [Lee et al. '13] Use mathematical programming to prepare for disease outbreak and medical catastrophes.
- Where should we place medical dispensing facilities, and how should we staff these in order to disseminate medication as quickly as possible to population?
- How should dispensing be scheduled?




2001 Anthrax letter sent to Senator T. Daschle

# Optimization in Disease Control

- In collaboration with the Center for Disease Control, [Lee et al. '13] develop decision support suite RealOpt
- Suite is being used by $\approx$ 6500 public health and emergency directors in the USA to design, place and staff medical dispensing centers
- In tests, throughput in medical dispensing centers increases by several orders of magnitude.

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- The following table gives precise requirements:

| Product | Machine 1 | Machine 2 | Skilled Labor | Unskilled Labor | Unit Sale Price |
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E.g.: producing a unit of product 3 requires 6h on machine 1, 5h on machine 2, 5h of skilled, and 7h of unskilled labour. It can be sold at $220 per unit.

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Restrictions:

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Formulate this as a mathematical program!

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- Objective function. A function of the variables that we would like to maximize/minimize.

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- Similarly, we may not use more than 500h of machine 2 time:

$$\Longrightarrow \quad 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500$$

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- Producing $x_i$ units of product $i \in \mathcal{P}$ requires

$$8x_1 + 5x_2 + 5x_3 + 6x_4$$

units of skilled labour, and this must not exceed $y_s$.

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- …and $y_s \leq 600$ as well as $y_u \leq 650$ as only limited amounts of labour can be purchased.

# WaterTech Model – Objective Function

- Revenue from sales:

$$300x_1 + 260x_2 + 220x_3 + 180x_4$$

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$$\begin{aligned} \text{maximize} \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 \\ & - 8y_s - 6y_u \end{aligned}$$

# WaterTech – Entire Model

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\begin{aligned}
\max \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t.} \quad & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
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\end{aligned}$$

Solution (via CPLEX): $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$, $y_u = 650$ of profit \$15433 + $\frac{1}{3}$.

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  - Similar: solution to word description is an assignment to the unknowns

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- It is easily checked that

$$x_1 = 10, x_2 = 50, x_3 = 0, x_4 = 20, y_s = y_u = 600$$

  is feasible for the mathematical program we wrote.

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- In the example, the map was simply the identity. This need not necessarily be the case in general!