Module 2: Linear Programs (Canonical Forms)

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Canonical form for $B = \{1, 2\}$

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$$\boxed{\begin{array}{ll} \max & \begin{pmatrix}-2 & 0 & 0 & 6\end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix}-1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1\end{pmatrix} x = \begin{pmatrix}1 \\ 1\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

Canonical form for $B = \{2, 3\}$

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Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

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Example

$(1) \quad \bar{x} = (1, 2, 0, 0)^\top$ is feasible for both LPs.

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For any basis B, there exists (P') in canonical form for B such that

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(1) $\bar{x} = (1, 2, 0, 0)^\top$ is feasible for both LPs.

(2)
$$\boxed{\begin{array}{rcr} (0 \quad 0 \quad 2 \quad 4)\bar{x} & = & 2 \times 0 + 4 \times 0 = 0 \\ (-2 \quad 0 \quad 0 \quad 6)\bar{x} \quad +2 & = & -2 \times 1 + 6 \times 0 + 2 = 0 \end{array}}$$

Illustration with an Example

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$$\max \quad \underbrace{\begin{pmatrix}0 & 0 & 2 & 4\end{pmatrix}}_{c} x$$

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Question

How do we rewrite (P) in canonical form for basis $B = \{2, 3\}$?

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(P1) Replace $Ax = b$ by $A'x = b'$ with $A'_{\{2,3\}} = I$.

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Remarks

- $A'_B = I$.
- $Ax = b$ and $A'x = b'$ have the same set of solutions.

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$$\max \quad z = \underbrace{\begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix}}_{c^\top} x$$

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$$0 = -(y_1 \;\; y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + (y_1 \;\; y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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Remark

For any choice of y_1, y_2 and any feasible solution x,

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Remark

For any choice of y_1, y_2 and any feasible solution x,

objective value of x for old objective function $=$
objective value of x for new objective function

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Question

How do we choose y_1, y_2 such that $\bar{c}_2 = \bar{c}_3 = 0$?

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$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\ 2\end{pmatrix} = \begin{pmatrix}2\\ 0\end{pmatrix}$$

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$\Rightarrow$

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$\Rightarrow$

$$z = (-2 \quad 0 \quad 0 \quad 6) x + 2$$

Rewriting the Objective Function – General

$$\boxed{\begin{aligned} \max \quad & z = c^\top x \\ \text{s.t.} \quad & \\ & Ax = b \\ & x \geq \mathbb{0} \end{aligned}}$$

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For any choice of y and any feasible solution x,

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Proposition

Let B be a basis of A,

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