Module 2: Linear Programs (Formalizing the Simplex)

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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Consider $B = \{1, 4\}$.

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Consider $B = \{1, 4\}$.

- A_B is square and non-singular

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis

Finding an Optimal Solution

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

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$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Rightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$ $\Rightarrow$ LP is in canonical form for B

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$ $\Longrightarrow$ LP is in canonical form for B
- $\bar{x} = (2, 0, 0, 5)^\top$ is a the basic solution for B.

Finding an Optimal Solution

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Rightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$ $\Rightarrow$ LP is in canonical form for B
- $\bar{x} = (2, 0, 0, 5)^\top$ is a the basic solution for B.
- $\bar{x} \geq \mathbf{0}$

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

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$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$ $\Longrightarrow$ LP is in canonical form for B
- $\bar{x} = (2, 0, 0, 5)^\top$ is a the basic solution for B.
- $\bar{x} \geq \mathbf{0}$ $\Longrightarrow$ $\bar{x}$ is feasible, i.e., B is feasible

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$
\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x
$$

s.t.

$$
\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}
$$

$$
x_1, x_2, x_3, x_4 \geq 0
$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Question

How do we find a better feasible solution?

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

Set $x_k = t \geq 0$ as large as possible.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

Set $x_k = t \geq 0$ as large as possible.

Keep all other non-basic variables at 0.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

Set $x_k = t \geq 0$ as large as possible.

Keep all other non-basic variables at 0.

Pick $k = 2$. Set $x_2 = t \geq 0$.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

Set $x_k = t \geq 0$ as large as possible.

Keep all other non-basic variables at 0.

Pick $k = 2$. Set $x_2 = t \geq 0$.

Keep $x_3 = 0$.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x$$

$$\begin{array}{ll} \max & \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x \\ \text{s.t.} & \\ & \underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned} \begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0,\ x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned}
\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix}
\end{aligned}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned}
\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}
\end{aligned}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

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$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned}
\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}
\end{aligned}$$

$$\boxed{\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

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$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$\max \quad \underbrace{\begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 1 & 2 & 0\\ 0 & 1 & 1 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}2\\ 5\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix}x_1\\ x_4\end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix}2\\ 5\end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix}1\\ 1\end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Rightarrow \quad t \leq \frac{2}{1}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{2}{1}$$

$$x_4 = 5 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{5}{1}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{2}{1}$$

$$x_4 = 5 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{5}{1}$$

Thus, the largest possible $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\}$.

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{2}{1}$$

$$x_4 = 5 - t \geq 0 \quad \Rightarrow \quad t \leq \tfrac{5}{1}$$

Thus, the largest possible $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\}$.

The new feasible solution is $x = (0, 2, 0, 3)^\top$. It has value $2 > 0$.

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

$$\max \quad \begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix} x$$

s.t.

$$\begin{pmatrix}1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 5\end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

Question

For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix}1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 5\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

Question

For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$x_2 \neq 0 \Longrightarrow 2 \in B$

$$\max \quad (0 \quad 1 \quad 3 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

Question

For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$$x_2 \neq 0 \quad \Rightarrow \quad 2 \in B$$

$$x_4 \neq 0 \quad \Rightarrow \quad 4 \in B$$

$$
\begin{array}{ll}
\max & \begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix} x \\
\text{s.t.} & \\
& \begin{pmatrix}1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 5\end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

Question

For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$$
\begin{array}{lcl}
x_2 \neq 0 & \Rightarrow & 2 \in B \\
x_4 \neq 0 & \Rightarrow & 4 \in B
\end{array}
$$

As $|B| = 2$, $B = \{2, 4\}$.

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\max \quad (0 \quad 1 \quad 3 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$\Downarrow$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

$$\begin{aligned} \max \quad & (0 \quad 1 \quad 3 \quad 0)x \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\begin{aligned} \max \quad & (-1 \quad 0 \quad 1 \quad 0)x + 2 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1,4\}$ is a feasible basis

Canonical form for $\{1,4\}$

$$\Downarrow$$

$$\begin{array}{ll} \max & \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

NEW

$\{2,4\}$ is a feasible basis

Canonical form for $\{2,4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

$$\begin{aligned} \max \quad & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

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$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\begin{aligned} \max \quad & \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x + 2 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

- 2 entered the basis.

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$\Downarrow$

$$\begin{array}{ll} \max & \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

- 2 entered the basis.
- 1 left the basis.

$$\begin{aligned} \max \quad & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

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Canonical form for $\{1, 4\}$

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$$\begin{aligned} \max \quad & \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x + 2 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

- 2 entered the basis.
- 1 left the basis.

WHY?

$$\max \quad \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$$\begin{array}{ll} \max & (0 \quad 1 \quad 3 \quad 0)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

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Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

⇒ 2 enters the basis

$$\begin{array}{ll} \max & (0 \quad 1 \quad 3 \quad 0)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

$\Rightarrow$ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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Pick $2 \notin B$ and set $x_2 = t \geq 0$.

$\Rightarrow$ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\} = 2$.

$$\begin{array}{ll} \max & (0 \quad 1 \quad 3 \quad 0)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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Canonical form for $\{1, 4\}$

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➡ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $t = \min \left\{ \frac{2}{1}, \frac{5}{1} \right\} = 2$.

➡ $x_1 = 0$ and 1 leaves the basis

$$\begin{array}{ll} \max & (0 \quad 1 \quad 3 \quad 0)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

➡ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\} = 2$.

➡ $x_1 = 0$ and 1 leaves the basis

The NEW basis is $\{2, 4\}$.

Example – Continued

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$$x_3 = t$$

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

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Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ➡ 3 enters the basis

Example – Continued

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$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ⟹ 3 enters the basis

Pick $x_B = b - tA_k$:

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ $\Longrightarrow$ 3 <u>enters</u> the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

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$t = \min\left\{\frac{2}{2}, -\right\} = 2$ thus $x_2 = 0$

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ $\Rightarrow$ 3 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$t = \min\left\{\frac{2}{2}, -\right\} = 2$ thus $x_2 = 0$ $\Rightarrow$ 2 leaves the basis

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ $\Longrightarrow$ 3 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$t = \min \left\{ \frac{2}{2}, - \right\} = 2$ thus $x_2 = 0$ $\Longrightarrow$ 2 leaves the basis

The NEW basis is $B = \{3, 4\}$.

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix} 0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\max \quad \underbrace{\begin{pmatrix} -1.5 & -0.5 & 0 & 0 \end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix} 0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

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$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

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Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

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$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix}0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1 \\ 4\end{pmatrix}}_{b}$$

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Canonical form for B

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: ???

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$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

Let x be a feasible solution.

$$\max \quad \underbrace{\begin{pmatrix}-1.5 & -0.5 & 0 & 0\end{pmatrix}}_{c} x + 3$$

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$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

Let x be a feasible solution. Then

$$\underbrace{(-1.5,\ 0.5,\ 0,\ 0)}_{\leq \mathbf{0}} \underbrace{x}_{\geq \mathbf{0}} + 3$$

$$\max \quad \underbrace{\begin{pmatrix} -1.5 & -0.5 & 0 & 0 \end{pmatrix}}_{c} x + 3$$

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: $???$

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

Let x be a feasible solution. Then

$$\underbrace{(-1.5,\ 0.5,\ 0,\ 0)}_{\leq \mathbf{0}} \underbrace{x}_{\geq \mathbf{0}} + 3 \leq 3.$$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

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$\{1, 4, 5\}$ is a feasible basis

Canonical form for $\{1, 4, 5\}$

Another Example

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

Another Example

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Canonical form for $\{1, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$$x_3 = t$$

Another Example

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ➡ 3 <u>enters</u> the basis

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

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$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

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Canonical form for $\{1, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t \Rightarrow$ 3 enters the basis

Pick $x_B = b - tA_k$:

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

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Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - t \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

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$x_3 = t$ $\Rightarrow$ 3 <u>enters</u> the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - t \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

$$t = \min\left\{\tfrac{1}{1}, -, -\right\} = 1$$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 4, 5\}$ is a feasible basis

Canonical form for $\{1, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ $\Rightarrow$ 3 enters the basis

Pick $x_B = b - tA_k$:

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$t = \min\left\{\frac{1}{1}, -, -\right\} = 1$ thus $x_1 = 0$ ➡ 1 <u>leaves</u> the basis

The NEW basis is $B = \{3, 4, 5\}$.

$$\max \quad \begin{pmatrix} -3 & 2 & 0 & 0 & 0 \end{pmatrix} x + 3$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$$\begin{array}{ll} \max & (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

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Canonical form for $\{3, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$$x_2 = t$$

$$\max \quad (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

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Canonical form for $\{3, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_2 = t$ ➡ 2 enters the basis

$$\begin{array}{ll} \max & (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$\{3, 4, 5\}$ is a feasible basis

Canonical form for $\{3, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_2 = t$ $\Rightarrow$ 2 <u>enters</u> the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} - t \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\max \quad \begin{pmatrix} -3 & 2 & 0 & 0 & 0 \end{pmatrix} x + 3$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

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Choose $t =$???

$$
\begin{aligned}
\max \quad & (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3 \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

$\{3, 4, 5\}$ is a feasible basis

Canonical form for $\{3, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_2 = t \quad \Longrightarrow \quad$ 2 <u>enters</u> the basis

Pick $x_B = b - tA_k$:

$$
\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} - t \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}
$$

Choose $t =???$

Claim

The linear program is unbounded.

$$\max \quad z = \begin{pmatrix} -3 & 2 & 0 & 0 & 0 \end{pmatrix} x + 3$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_2 = t$$

$$x_1 = 0$$

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} - t \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\max \quad z = \begin{pmatrix}-3 & 2 & 0 & 0 & 0\end{pmatrix} x + 3$$

s.t.

$$\begin{pmatrix}1 & -2 & 1 & 0 & 0\\ 3 & -1 & 0 & 1 & 0\\ 2 & 0 & 0 & 0 & 1\end{pmatrix} x = \begin{pmatrix}1\\4\\4\end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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$$\begin{pmatrix}x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}1\\4\\4\end{pmatrix} - t\begin{pmatrix}-2\\-1\\0\end{pmatrix}$$

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Claim

The linear program is unbounded.

Proof

$$x(t) = \begin{pmatrix} 0 \\ t \\ 1 + 2t \\ 4 + t \\ 4 \end{pmatrix} =$$

$$\max \quad z = (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

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The linear program is unbounded.

Proof

$$x(t) = \begin{pmatrix} 0 \\ t \\ 1+2t \\ 4+t \\ 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 4 \end{pmatrix}}_{=\bar{x}} + t \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}}_{=r}$$

$$\max \quad z = \begin{pmatrix}-3 & 2 & 0 & 0 & 0\end{pmatrix} x + 3$$

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- $x(t)$ is feasible for all $t \geq 0$.

$$\max \quad z = (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3$$

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- $x(t)$ is feasible for all $t \geq 0$.
- $z \to \infty$ when $t \to \infty$.

$$\max \quad z = (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3$$

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- $x(t)$ is feasible for all $t \geq 0$.
- $z \to \infty$ when $t \to \infty$.

($\bar{x}, r$: certificate of unboundedness.)

The Simplex Algorithm

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$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

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<u>INPUT:</u>

The Simplex Algorithm

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

<u>Input:</u> a feasible basis B.

The Simplex Algorithm

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

INPUT: a feasible basis B.

OUTPUT:

The Simplex Algorithm

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

<u>INPUT:</u> a feasible basis B.

<u>OUTPUT:</u> an optimal solution OR

The Simplex Algorithm

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}$$

INPUT: a feasible basis B.

OUTPUT: an optimal solution OR it detects that the LP is unbounded.

The Simplex Algorithm

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

<u>INPUT:</u> a feasible basis B.

<u>OUTPUT:</u> an optimal solution OR it detects that the LP is unbounded.

Step 1. Rewrite in canonical form for the basis B.

The Simplex Algorithm

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

<u>INPUT:</u> a feasible basis B.

<u>OUTPUT:</u> an optimal solution OR it detects that the LP is unbounded.

Step 1. Rewrite in canonical form for the basis B.

Step 2. Find a better basis B or get required outcome.

The Simplex Algorithm

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

<u>INPUT:</u> a feasible basis B.

<u>OUTPUT:</u> an optimal solution OR it detects that the LP is unbounded.

Step 1. Rewrite in canonical form for the basis B.

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Trying to Find a Better Basis

$$\boxed{\begin{aligned} \max \quad & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} \quad & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{aligned}}$$

Trying to Find a Better Basis

$$\begin{array}{|ll|}\hline \max & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \\ \hline \end{array}$$

B is a feasible basis,

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B is a feasible basis, $N = \{j \notin B\}$

Trying to Find a Better Basis

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\end{array}
}
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Canonical form for B

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Canonical form for B

$\bar{x}$ is a basic solution

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B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

$\bar{x}$ is a basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

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Pick $x_B = b - tA_k$.

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Choose $t = \min\left\{\frac{b_i}{A_{ik}} : \text{for all } i \text{ such that } A_{ik} > 0\right\}$.

Trying to Find a Better Basis

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\begin{array}{ll}
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Let x_r be a basic variable forced to 0.

Trying to Find a Better Basis

$$\boxed{\begin{array}{ll} \max & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}}$$

B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

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If $A_k \leq \mathbf{0}$, then STOP. The LP is unbounded.

Choose $t = \min\left\{ \frac{b_i}{A_{ik}} : \text{for all } i \text{ such that } A_{ik} > 0 \right\}$.

Let x_r be a basic variable forced to 0.

The new basis is obtained by having k enter and r leave.

$$\begin{array}{|ll|}\hline \max & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \\ \hline \end{array}$$

B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

$\bar{x}$ basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

$$
\begin{array}{ll}
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& x_B + A_N x_N = b \\
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Canonical form for B

$\bar{x}$ basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

Proof

$$\begin{array}{ll} \max & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

$\bar{x}$ basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

Proof

$\bar{x}_B = b$, $\bar{x}_N = \mathbf{0}$.

$$
\begin{array}{ll}
\max & z = c_N^\top x_N + \bar{z} \\
\text{s.t.} & \\
& x_B + A_N x_N = b \\
& x \geq \mathbf{0}
\end{array}
$$

B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

$\bar{x}$ basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

Proof

$\bar{x}_B = b$, $\bar{x}_N = \mathbf{0}$.

$\bar{x}$ has value $z = c_N^\top \bar{x}_N + \bar{z} = \bar{z}$.

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Let us see an example...

$$\max \quad \begin{pmatrix} 0 & 0 & 2 & 3 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 2\}$ is a feasible basis

Canonical form for $\{1, 2\}$

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$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} - t \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \text{and} \quad t = \min\left\{\tfrac{6}{2}, \tfrac{12}{4}\right\} = 3$$

Pick $r \in B$ such that $x_r = 0$:

Choices $r = 1$ OR $r = 2$.

Bland's rule says

$$\max \quad \begin{pmatrix} 0 & 0 & 2 & 3 \end{pmatrix} x$$

s.t.

$$\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 2\}$ is a feasible basis

Canonical form for $\{1, 2\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

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The NEW basis is $B = \{3, 4\}$.

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  To do: Find a procedure to find a feasible basis.