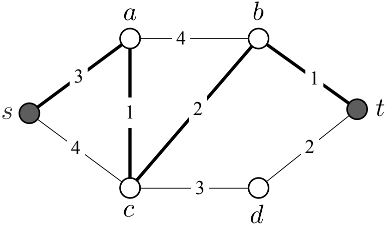
Module 3: Duality through examples

Recap: Shortest Paths

In an instance of the shortest path problem, we are given

- a graph $G = (V, E)$, a non-negative length c_e for each edge $e \in E$, and
- a pair of vertices s and t in V.



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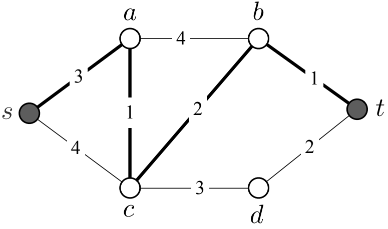

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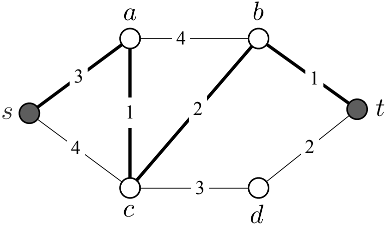
Recall: an s, t-path is a sequence

$$P := u_1u_2, u_2u_3, \ldots, u_{k-1}u_k$$

where

- $u_iu_{i+1} \in E$ for all i, and
- $u_1 = s$, $u_k = t$, and $u_i \neq u_j$ for all $i \neq j$.



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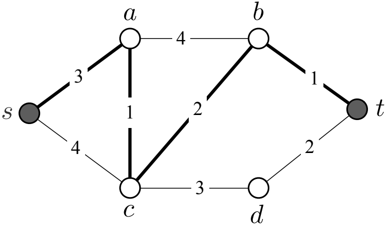


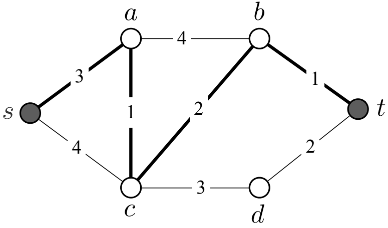
Its length is given by

$$c(P) = c_{u_1u_2} + c_{u_2u_3} + \ldots + c_{u_{k-1}u_k}$$

In the example, we see by inspection that

$$P = sa, ac, cb, bt$$

is a shortest path and that its length is 9.



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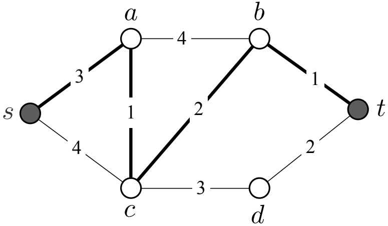

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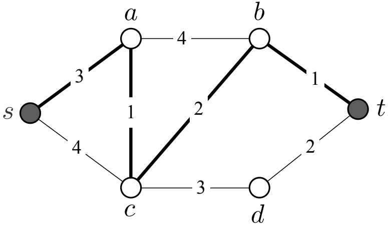

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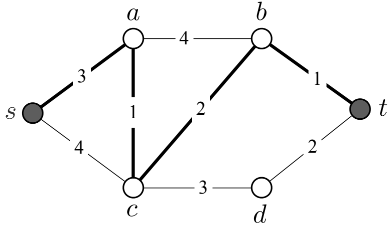


Question

1. Given a shortest-path instance and a candidate shortest s,t-path P, is there a short proof of its optimality?

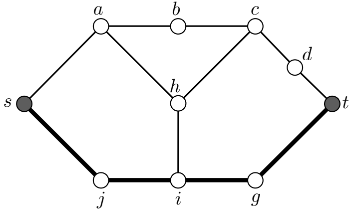
2. How can we find a shortest s,t-path?

We will answer both questions in this module. This lecture focus on question 1.

Shortest Paths: Finding an Intuitive Lower Bound

Cardinality Case

To make our lives easier, we will first consider the cardinality special case of the shortest path problem.



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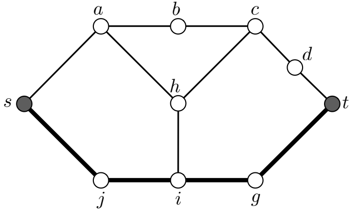

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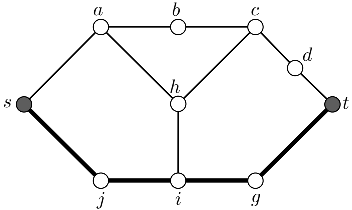

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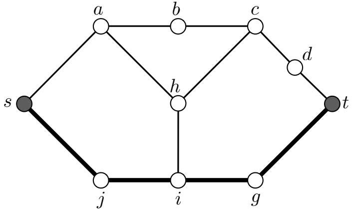

Example: In the diagram above, one easily sees that

$$P = sj, ji, ig, gt$$

is a shortest s, t-path.

How can we prove this fact?

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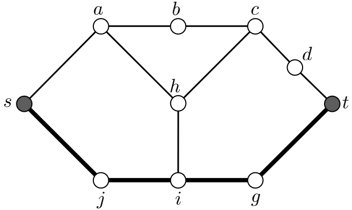

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How can we prove this fact?
$\longrightarrow$ The answer lies in s,t-cuts!

s, t-cuts

Definition

For $U \subseteq V$, we define

$$\delta(U) = \{uv \in E \,:\, u \in U, v \notin U\}$$

and call it an s, t-cut if $s \in U$, and $t \notin U$.

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Example

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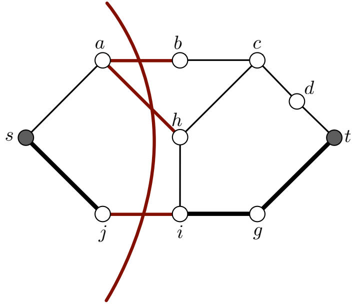

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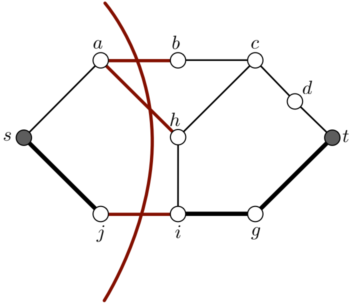

Recall:

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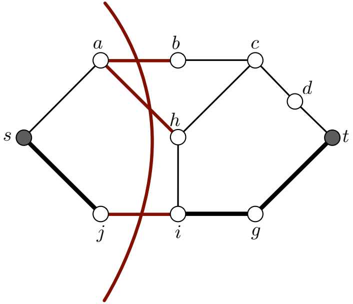
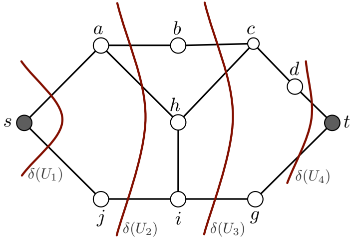


Recall:

- If P is an s, t-path and $\delta(U)$ an s, t-cut, then P contains an edge of $\delta(U)$.
- If $S \subseteq E$ contains an edge from every s, t-cut, then S contains an s, t-path.

From Cuts to Lower-Bounds

The example on the right shows 4 s,t-cuts, $\delta(U_1),\delta(U_2),\delta(U_3),\delta(U_4)$.



$$
\begin{aligned}
\delta(U_1) &= \{sa, sj\} \\
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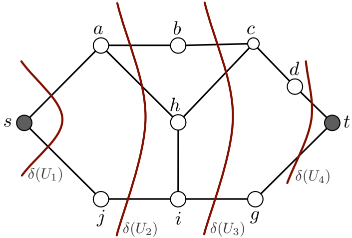
The example on the right shows 4 s,t-cuts, $\delta(U_1), \delta(U_2), \delta(U_3), \delta(U_4)$.

Two important notes:

(1) $\delta(U_i) \cap \delta(U_j) = \emptyset$ for $i \neq j$ and



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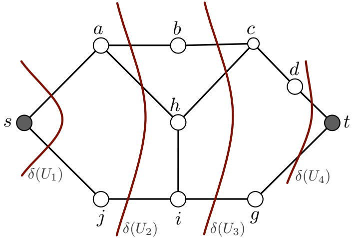

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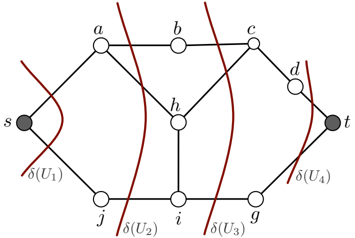
$\longrightarrow$ Every s,t-path must have at least 4 edges.



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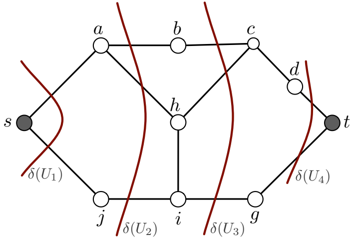

$\longrightarrow$ Every s,t-path must have at least 4 edges.
$\longrightarrow$ sj, ji, ig, gt is a shortest s,t-path!

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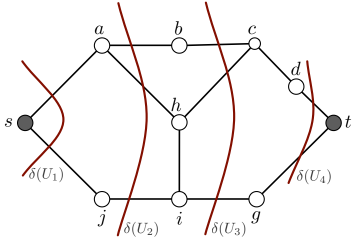

Question

Notice: hi is not in any of the $\delta(U_i)$. Does this mean that hi is not on any shortest s,t-path?

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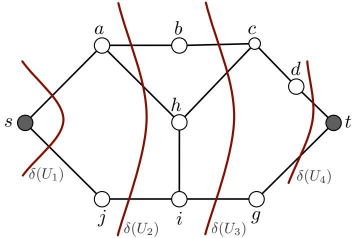

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An s,t-path that contains hi must also contain an edge from each of the s,t-cuts $\delta(U_i)$.

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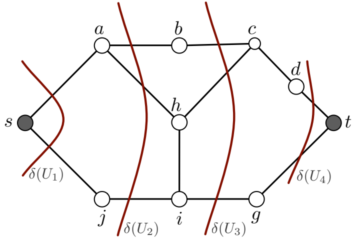

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An s,t-path that contains hi must also contain an edge from each of the s,t-cuts $\delta(U_i)$. $\longrightarrow$ It must contain at least 5 edges!

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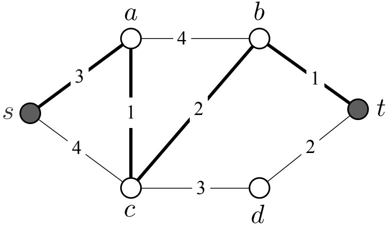
Back to the General Case

In general instances, we assign a non-negative width y_U to every s,t-cut $\delta(U)$.



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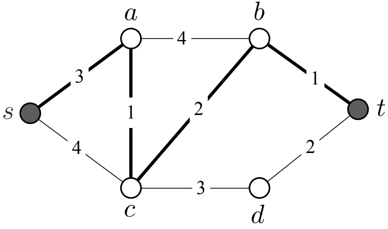

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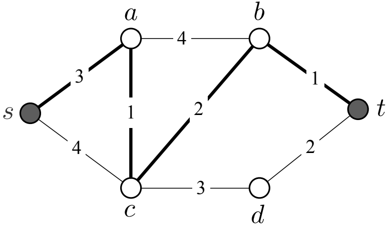

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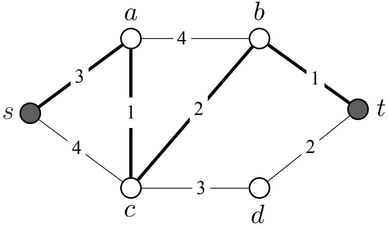

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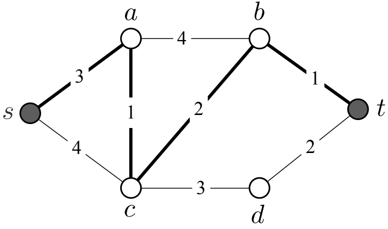

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Using math: y is feasible if for all e

$$\sum (y_U \,:\, \delta(U)\ s,t\text{-cut and } e \in E) \leq c_e$$

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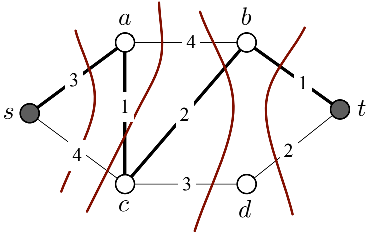
Consider the example on the right with 4 s,t-cuts.



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\begin{aligned}
U_1 &= \{s\} \\
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U_4 &= \{s, a, b, c, d\}
\end{aligned}
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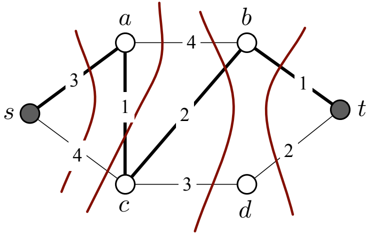
Back to the General Case

Consider the example on the right with 4 s,t-cuts.

The width assignment

$$\begin{aligned} y_{U_1} &= 3 \\ y_{U_2} &= 1 \\ y_{U_3} &= 2 \\ y_{U_4} &= 1 \end{aligned}$$

is easily checked to be feasible.



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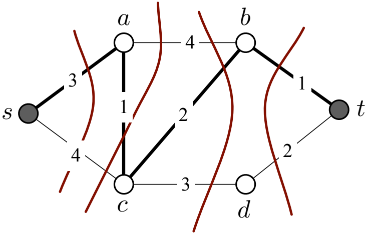
Proposition

If y is a feasible width assignment, then any s,t-path must have length at least

$$\sum(y_U \,:\, U \;\; s,t\text{-cut}).$$



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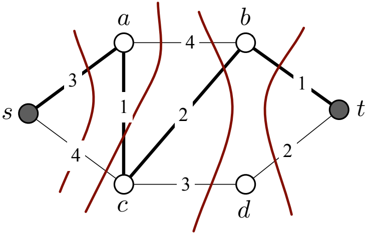

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$$y_{U_1} + y_{U_2} + y_{U_3} + y_{U_4} = 7$$

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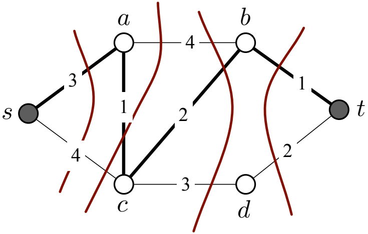

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Proof: Consider an s,t-path P. It follows that

$$c(P) \quad = \quad \sum(c_e \,:\, e \in P)$$

$\square$

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$\longrightarrow$ Path sa, ac, cb, bt is a shortest path!

Proof: Consider an s,t-path P. It follows that

$$\begin{aligned} c(P) &= \sum(c_e \,:\, e \in P) \\ &\geq \sum\left(\sum(y_U \,:\, e \in \delta(U)) \,:\, e \in P\right) \end{aligned}$$

where the first inequality follows from the feasibility of y.

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Back to the General Case

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If y is a feasible width assignment, then any s,t-path must have length at least

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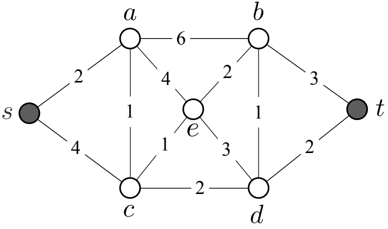
One More Example

Question: Can you spot a shortest s, t-path?



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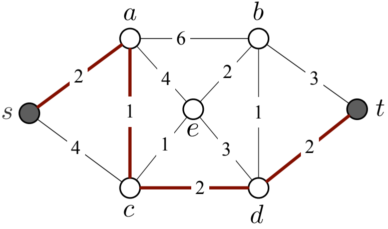

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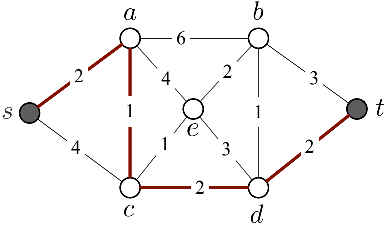

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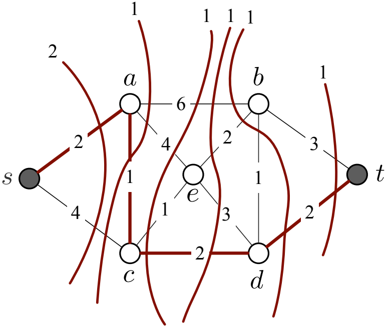
Question: Can you spot a shortest s, t-path?
$\longrightarrow$ $P = sa, ac, cd, dt$ of length 7.

Question: Can you prove your guess?
$\longrightarrow$ Yes! There is a feasible dual width assignment of value 7:

$$
\begin{aligned}
y_{\{s\}} &= 2 \\
y_{\{s,a\}} &= 1 \\
y_{\{s,a,c\}} &= 1 \\
y_{\{s,a,c,e\}} &= 1 \\
y_{\{s,a,c,d,e\}} &= 1 \\
y_{\{s,a,b,c,d,e\}} &= 1
\end{aligned}
$$



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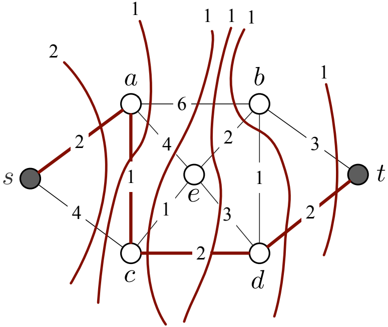
(A) In an instance with a shortest path, can we always find feasible widths to prove optimality?



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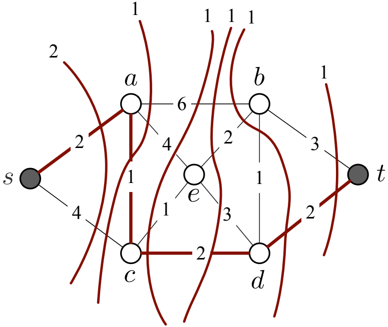
(A) In an instance with a shortest path, can we always find feasible widths to prove optimality?

(B) If so, how do we find a path and these widths?



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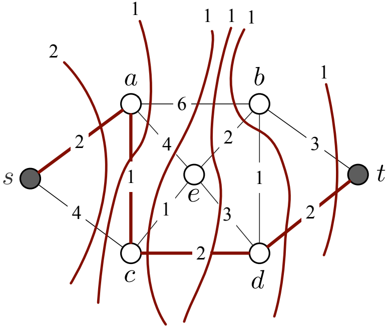

We will answer (A) affirmatively, and provide an efficient algorithm for (B) shortly.

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  for all $e \in E$.

- If y is a feasible width assignment and P an s, t-path, then

$$c(P) \geq \sum y_U$$