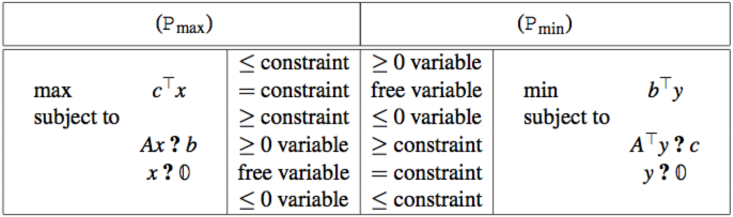
Module 4: Duality Theory (Strong Duality)

Recap: Weak Duality



(P_{max})		(P_{min})	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \, ? \, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \, ? \, c$
$x \, ? \, \mathbb{0}$	free variable	$=$ constraint	$y \, ? \, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Last lecture: we described a method to construct the dual of a general linear program.

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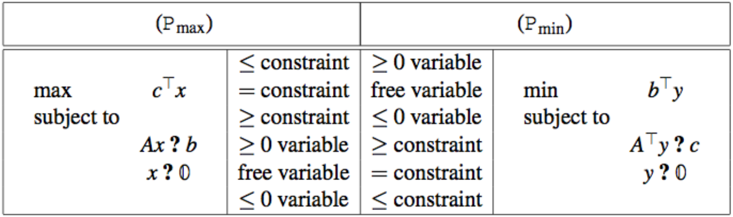

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
max $c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} \mathbb{0}$
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	≥ 0 variable	$\geq$ constraint	
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E.g.: consider the primal LP, (P), on the right

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

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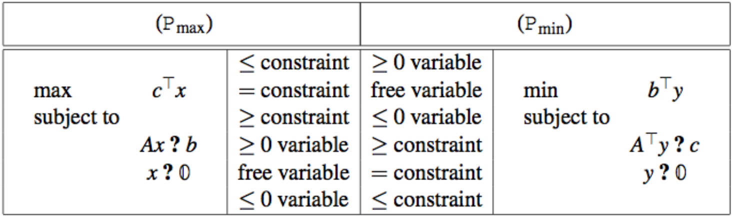

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Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right – a max LP that falls in the left (P$_{\max}$) part of the table.

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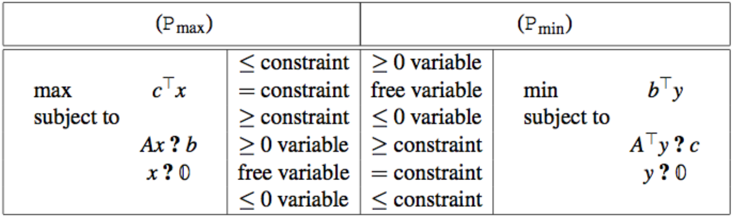

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Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right – a max LP that falls in the left ($P_{\max}$) part of the table.

$\longrightarrow$ The dual of (P) is a min LP.

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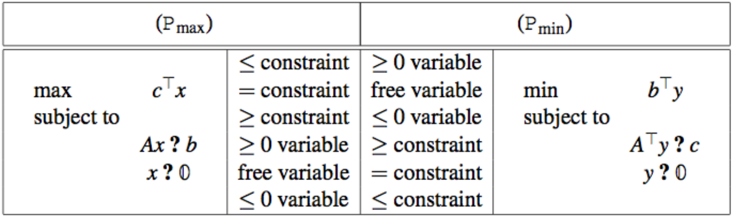

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$$\text{s.t.} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \ ?\ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

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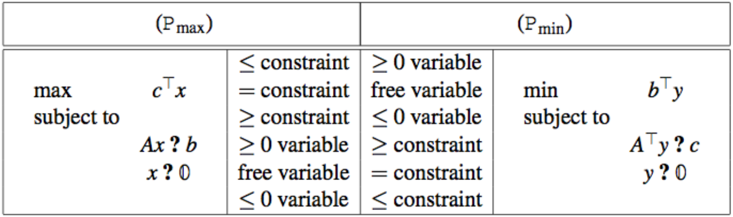

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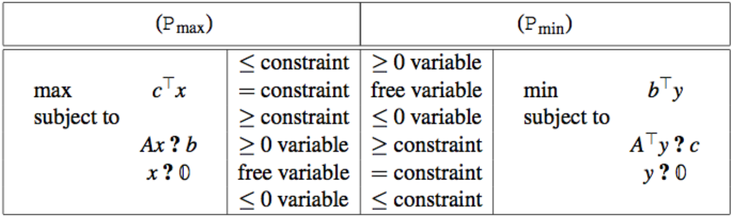

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Weak Duality Theorem

if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D),

$$\implies \quad c^T \bar{x} \le b^T \bar{y}$$

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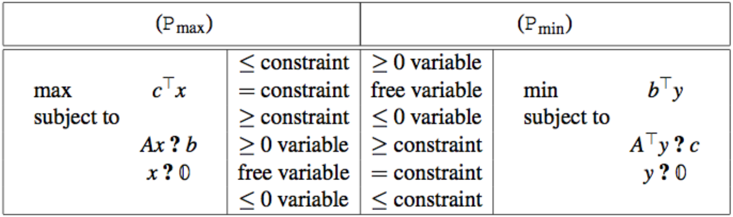
Weak Duality Theorem

if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D),

$$\Longrightarrow \quad c^T\bar{x} \le b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then both $\bar{x}$ and $\bar{y}$ are optimal.

This Lecture: Strong Duality



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Question

Can we always find feasible solutions $\bar{x}$ and $\bar{y}$ to a primal-dual pair, $(\mathrm{P}_{\max})$, $(\mathrm{P}_{\min})$, such that $c^T\bar{x} = b^T\bar{y}$?

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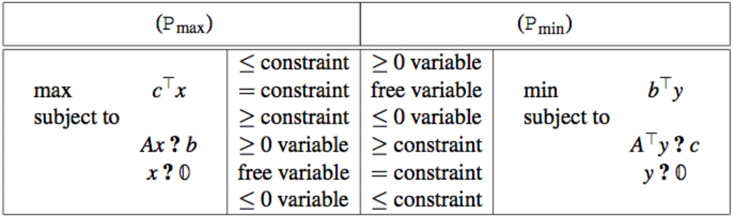

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Can we always find feasible solutions $\bar{x}$ and $\bar{y}$ to a primal-dual pair, $(\mathrm{P}_{\max})$, $(\mathrm{P}_{\min})$, such that $c^T\bar{x} = b^T\bar{y}$?

Strong Duality Theorem

If $(\mathrm{P}_{\max})$ has an optimal solution $\bar{x}$, then $(\mathrm{P}_{\min})$ has an optimal solution $\bar{y}$ such that $c^T\bar{x} = b^T\bar{y}$.

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

$$\begin{aligned} \max\ & c^T x \qquad\qquad \text{(P)} \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

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We can rewrite (P) for basis B:

$$\begin{aligned} \max\ & z = \bar{y}^T b + \bar{c}^T x && \text{(P')} \\ \text{s.t. } & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\ & x \geq \mathbb{0} \end{aligned}$$

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where:

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and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$

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Recall that (P) and (P') are equivalent!

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$$\begin{aligned} c^T \bar{x} &= \bar{y}^T b + \bar{c}^T \bar{x} \\ &= \bar{y}^T b + \bar{c}_N^T \bar{x}_N \end{aligned}$$

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$$\begin{aligned} c^T \bar{x} &= \bar{y}^T b + \bar{c}^T \bar{x} \\ &= \bar{y}^T b + \bar{c}_N^T \bar{x}_N \\ &= b^T \bar{y} \end{aligned}$$

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Goal: Show that $\bar{y}$ is dual feasible.

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Strong Duality – for LPs in SEF

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Equivalently, $A^T \bar{y} \geq c$,

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Subtly different version via previous results:

Strong Duality Theorem – Feasibility Version

Let (P) and (D) be primal-dual pair of LPs. If both are feasible, then both have optimal solutions of the same objective value.

Possible Outcomes of Primal-Dual Pair (P), (D)

(P) (D)	optimal solution	unbounded	infeasible
optimal solution			
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Possible Outcomes of Primal-Dual Pair (P), (D)

(P) / (D)	optimal solution	unbounded	infeasible
optimal solution	possible ①		
unbounded			possible ⑥
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- ①, ⑥, and ⑧ many examples exist

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- ③, ⑦ follow directly from Strong Duality
- I'll leave ⑨ for you to do as an exercise!

$$\max\ c^Tx \qquad \text{(P)}$$
$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

$$\min\ b^Ty \qquad \text{(D)}$$
$$\text{s.t. } A^Ty \geq c$$

Recap

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