Module 5: Integer Programs (IP versus LP)

LP versus IP

LP versus IP

Linear programming	Integer programming
Can solve very large instances	

LP versus IP

Linear programming	Integer programming
Can solve very large instances	
Algorithms exist that are guaranteed to be fast	

LP versus IP

Linear programming	Integer programming
Can solve very large instances	
Algorithms exist that are guaranteed to be fast	
Short certificate of infeasibility (Farka's Lemma)	

LP versus IP

Linear programming	Integer programming
Can solve very large instances	
Algorithms exist that are guaranteed to be fast	
Short certificate of infeasibility (Farka's Lemma)	
Short certificate of optimality (Strong Duality)	

LP versus IP

LINEAR PROGRAMMING	INTEGER PROGRAMMING
Can solve very large instances	
Algorithms exist that are guaranteed to be fast	
Short certificate of infeasibility (Farka's Lemma)	
Short certificate of optimality (Strong Duality)	
The only possible outcomes are infeasible, unbounded, or optimal	

LP versus IP

Linear programming	Integer programming
Can solve very large instances	Some small instances cannot be solved
Algorithms exist that are guaranteed to be fast	
Short certificate of infeasibility (Farka's Lemma)	
Short certificate of optimality (Strong Duality)	
The only possible outcomes are infeasible, unbounded, or optimal	

LP versus IP

Linear programming	Integer programming
Can solve very large instances	Some small instances cannot be solved
Algorithms exist that are guaranteed to be fast	No fast algorithm exists
Short certificate of infeasibility (Farka's Lemma)	
Short certificate of optimality (Strong Duality)	
The only possible outcomes are infeasible, unbounded, or optimal	

Remark

We cannot PROVE an algorithm that is guaranteed to be fast does not exist, but we can show that it is "highly unlikely".

LP versus IP

Linear programming	Integer programming
Can solve very large instances	Some small instances cannot be solved
Algorithms exist that are guaranteed to be fast	No fast algorithm exists
Short certificate of infeasibility (Farka's Lemma)	Does not always exist
Short certificate of optimality (Strong Duality)	
The only possible outcomes are infeasible, unbounded, or optimal	

Remark

We cannot PROVE that sometimes there is no short certificate of infeasibility, but we can show that it is "highly unlikely".

LP versus IP

Linear programming	Integer programming
Can solve very large instances	Some small instances cannot be solved
Algorithms exist that are guaranteed to be fast	No fast algorithm exists
Short certificate of infeasibility (Farka's Lemma)	Does not always exist
Short certificate of optimality (Strong Duality)	Does not always exist
The only possible outcomes are: infeasible, unbounded, or optimal	

Remark

We cannot PROVE that sometimes there is no short certificate of optimality, but we can show that it is "highly unlikely".

LP versus IP

Linear programming	Integer programming
Can solve very large instances	Some small instances cannot be solved
Algorithms exist that are guaranteed to be fast	No fast algorithm exists
Short certificate of infeasibility (Farka's Lemma)	Does not always exist
Short certificate of optimality (Strong Duality)	Does not always exist
The only possible outcomes are infeasible, unbounded, or optimal	Can have other outcomes

LP versus IP

Linear programming	Integer programming
Can solve very large instances	Some small instances cannot be solved
Algorithms exist that are guaranteed to be fast	No fast algorithm exists
Short certificate of infeasibility (Farka's Lemma)	Does not always exist
Short certificate of optimality (Strong Duality)	Does not always exist
The only possible outcomes are infeasible, unbounded, or optimal	Can have other outcomes

Let us look at an example...

A Bad Example

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{l} \max \quad x_1 - \sqrt{2}x_2 \\ \text{s.t.} \\ \qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ \qquad x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{l} \max \quad x_1 - \sqrt{2}x_2 \\ \text{s.t.} \\ \qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ \qquad x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

- It is feasible.

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

- It is feasible. ✓

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

- It is feasible. ✓
- 0 is an upper bound.

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{l} \max \quad x_1 - \sqrt{2}x_2 \\ \text{s.t.} \\ \qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ \qquad x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

- It is feasible. ✓
- 0 is an upper bound. ✓

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{l} \max \quad x_1 - \sqrt{2}x_2 \\ \text{s.t.} \\ \qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ \qquad x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

- It is feasible. ✓
- 0 is an upper bound. ✓
- It has no optimal solution.

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{l} \max \quad x_1 - \sqrt{2}x_2 \\ \text{s.t.} \\ \qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ \qquad x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

- It is feasible. ✓
- 0 is an upper bound. ✓
- It has no optimal solution. ???

$$
\boxed{
\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}
}
$$

No optimal solution

$$\boxed{\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2.

$$\boxed{\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \le & \sqrt{2}x_2 \\ x_1, x_2 & \ge & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}}$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

$$\boxed{\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

We will show:

$$
\boxed{
\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \le & \sqrt{2}x_2 \\ x_1, x_2 & \ge & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}
}
$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

We will show:

Claim 1. x_1', x_2' are feasible.

$$
\boxed{\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}}
$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

We will show:

Claim 1. x_1', x_2' are feasible.

Claim 2. x_1', x_2' has larger value than x_1, x_2.

$$\max \quad x_1 - \sqrt{2}x_2$$

s.t.

$$\begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array}$$

x_1, x_2 integer

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x'_1 = 2x_1 + 2x_2 \qquad x'_2 = x_1 + 2x_2$$

We will show:

Claim 1. x'_1, x'_2 are feasible.

Claim 2. x'_1, x'_2 has larger value than x_1, x_2.

⇒

contradiction !!!

$$
\boxed{
\begin{array}{l}
\max \quad x_1 - \sqrt{2}x_2 \\
\text{s.t.} \\
\qquad \begin{array}{rcl}
x_1 & \leq & \sqrt{2}x_2 \\
x_1, x_2 & \geq & 1
\end{array} \\
\quad x_1, x_2 \text{ integer}
\end{array}
}
$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x'_1 = 2x_1 + 2x_2 \qquad x'_2 = x_1 + 2x_2$$

Claim 1. x'_1, x'_2 are feasible.

Proof

$$
\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}
$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$
\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}
$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$x_1' \overset{?}{\leq} \sqrt{2}x_2'$$

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & x_1 \leq \sqrt{2}x_2 \\ & x_1, x_2 \geq 1 \\ & x_1, x_2 \text{ integer} \end{array}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$\begin{aligned} x_1' &\overset{?}{\leq} \sqrt{2}x_2' \qquad \Longleftrightarrow \\ 2x_1 + 2x_2 &\overset{?}{\leq} \sqrt{2}\,(x_1 + 2x_2) \end{aligned}$$

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{lcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$\begin{array}{lll} x_1' & \stackrel{?}{\leq} \sqrt{2}x_2' & \iff \\ 2x_1 + 2x_2 & \stackrel{?}{\leq} \sqrt{2}\,(x_1 + 2x_2) = \sqrt{2}x_1 + 2\sqrt{2}x_2 & \end{array}$$

$$\max \quad x_1 - \sqrt{2}x_2$$

s.t.

$$\begin{aligned} x_1 &\leq \sqrt{2}x_2 \\ x_1, x_2 &\geq 1 \end{aligned}$$

x_1, x_2 integer

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$\begin{aligned} x_1' &\overset{?}{\leq} \sqrt{2}x_2' && \iff \\ 2x_1 + 2x_2 &\overset{?}{\leq} \sqrt{2}\,(x_1 + 2x_2) = \sqrt{2}x_1 + 2\sqrt{2}x_2 && \iff \\ x_1(2 - \sqrt{2}) &\overset{?}{\leq} (2\sqrt{2} - 2)x_2 \end{aligned}$$

$$
\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}
$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$
\begin{array}{lll}
x_1' & \overset{?}{\leq} \sqrt{2}x_2' & \iff \\
2x_1 + 2x_2 & \overset{?}{\leq} \sqrt{2}\,(x_1 + 2x_2) = \sqrt{2}x_1 + 2\sqrt{2}x_2 & \iff \\
x_1(2 - \sqrt{2}) & \overset{?}{\leq} (2\sqrt{2} - 2)x_2 & \iff \\
x_1 & \overset{?}{\leq} \frac{2\sqrt{2}-2}{2-\sqrt{2}}x_2 &
\end{array}
$$

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{lll} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$\begin{array}{llr} x_1' & \stackrel{?}{\leq} \sqrt{2}x_2' & \iff \\ 2x_1 + 2x_2 & \stackrel{?}{\leq} \sqrt{2}\,(x_1 + 2x_2) = \sqrt{2}x_1 + 2\sqrt{2}x_2 & \iff \\ x_1(2 - \sqrt{2}) & \stackrel{?}{\leq} (2\sqrt{2} - 2)x_2 & \iff \\ x_1 & \stackrel{?}{\leq} \frac{2\sqrt{2}-2}{2-\sqrt{2}}x_2 = \sqrt{2}x_2 & \end{array}$$

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \le & \sqrt{2}x_2 \\ x_1, x_2 & \ge & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Proof

$x_1' = 2x_1 + 2x_2 \ge 1$ and $x_2' = x_1 + 2x_2 \ge 1$ ✓

$$\begin{array}{lll} x_1' & \overset{?}{\le} \sqrt{2}x_2' & \iff \\ 2x_1 + 2x_2 & \overset{?}{\le} \sqrt{2}\,(x_1 + 2x_2) = \sqrt{2}x_1 + 2\sqrt{2}x_2 & \iff \\ x_1(2-\sqrt{2}) & \overset{?}{\le} (2\sqrt{2}-2)x_2 & \iff \\ x_1 & \overset{?}{\le} \frac{2\sqrt{2}-2}{2-\sqrt{2}}x_2 = \sqrt{2}x_2 & \checkmark \end{array}$$

$$
\boxed{\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}}
$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

$$
\boxed{\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \le & \sqrt{2}x_2 \\ x_1, x_2 & \ge & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}}
$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Claim 2. x_1', x_2' has larger value than x_1, x_2.

$$
\boxed{\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{lll} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}}
$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Claim 2. $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$.

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Claim 2. $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$.

Proof

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{lcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Claim 2. $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$.

Proof

$$(2x_1 + 2x_2) - \sqrt{2}(x_1 + 2x_2)\sqrt{2} \overset{?}{>} x_1 - \sqrt{2}x_2$$

$$
\begin{array}{lll}
\max & x_1 - \sqrt{2}x_2 & \\
\text{s.t.} & & \\
& x_1 & \leq \sqrt{2}x_2 \\
& x_1, x_2 & \geq 1 \\
& x_1, x_2 \text{ integer} &
\end{array}
$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Claim 2. $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$.

Proof

$$(2x_1 + 2x_2) - \sqrt{2}(x_1 + 2x_2)\sqrt{2} \overset{?}{>} x_1 - \sqrt{2}x_2$$

Simplifying (takes a little work), we obtain

$$
\begin{array}{ll}
\max & x_1 - \sqrt{2}x_2 \\
\text{s.t.} & \\
& \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\
& x_1, x_2 \text{ integer}
\end{array}
$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Claim 2. $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$.

Proof

$$(2x_1 + 2x_2) - \sqrt{2}(x_1 + 2x_2)\sqrt{2} \overset{?}{>} x_1 - \sqrt{2}x_2$$

Simplifying (takes a little work), we obtain

$$\sqrt{2}x_2 \overset{?}{>} x_1$$

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{lcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim 1. x_1', x_2' are feasible.

Claim 2. $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$.

Proof

$$(2x_1 + 2x_2) - \sqrt{2}(x_1 + 2x_2)\sqrt{2} \stackrel{?}{>} x_1 - \sqrt{2}x_2$$

Simplifying (takes a little work), we obtain

$$\sqrt{2}x_2 \stackrel{?}{>} x_1$$

- $\geq$ since x_1, x_2 are feasible for (IP)

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x'_1 = 2x_1 + 2x_2 \qquad x'_2 = x_1 + 2x_2$$

Claim 1. x'_1, x'_2 are feasible.

Claim 2. $x'_1 - \sqrt{2}x'_2 > x_1 - \sqrt{2}x_2$.

Proof

$$(2x_1 + 2x_2) - \sqrt{2}(x_1 + 2x_2)\sqrt{2} \overset{?}{>} x_1 - \sqrt{2}x_2$$

Simplifying (takes a little work), we obtain

$$\sqrt{2}x_2 \overset{?}{>} x_1$$

- $\geq$ since x_1, x_2 are feasible for (IP)
- $>$ otherwise $\sqrt{2} = \frac{x_1}{x_2}$ but $\sqrt{2}$ is not a rational number

Bad News/Good News

Bad News/Good News

Bad News:

Bad News/Good News

Bad News:

- IPs are hard to solve.

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.
- Commercial codes solve many of these problems routinely.

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.
- Commercial codes solve many of these problems routinely.
- Some of the theory developed for LPs extends to IPs.

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.
- Commercial codes solve many of these problems routinely.
- Some of the theory developed for LPs extends to IPs.

This lecture will show:

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.
- Commercial codes solve many of these problems routinely.
- Some of the theory developed for LPs extends to IPs.

This lecture will show:

Integer Programming can, in principle, be reduced to Linear Programming.

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.
- Commercial codes solve many of these problems routinely.
- Some of the theory developed for LPs extends to IPs.

This lecture will show:

Integer Programming can, in principle, be reduced to Linear Programming.

Remark

This will NOT give us a practical procedure to solve IPs,

Bad News/Good News

Bad News:

- IPs are hard to solve.
- Theory for IPs is harder than for LPs.
- Results are not as powefull.

Good News:

- IPs can formulate a huge number of practical problems.
- Commercial codes solve many of these problems routinely.
- Some of the theory developed for LPs extends to IPs.

This lecture will show:

Integer Programming can, in principle, be reduced to Linear Programming.

Remark

This will NOT give us a practical procedure to solve IPs,

but it will suggest a strategy.

Definition

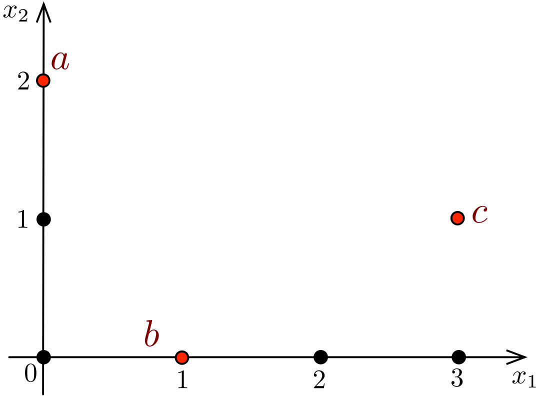
Let C be a subset of $\Re^n$.

Definition

Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.

Definition

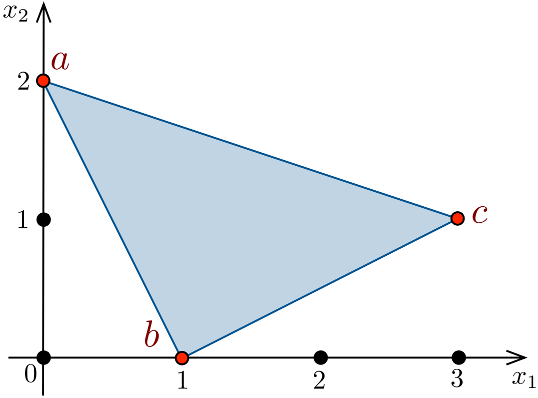
Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.



$$C = \{a, b, c\}$$

Definition

Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.

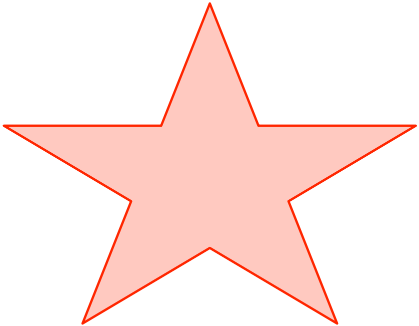


$C = \{a, b, c\}$

Convex hull of C

Definition

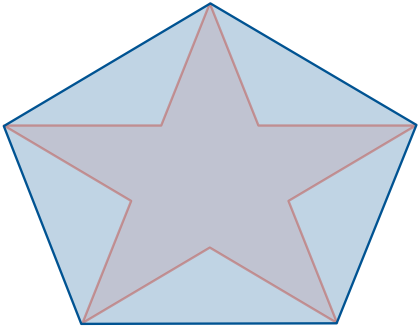
Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.



C

Definition

Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.

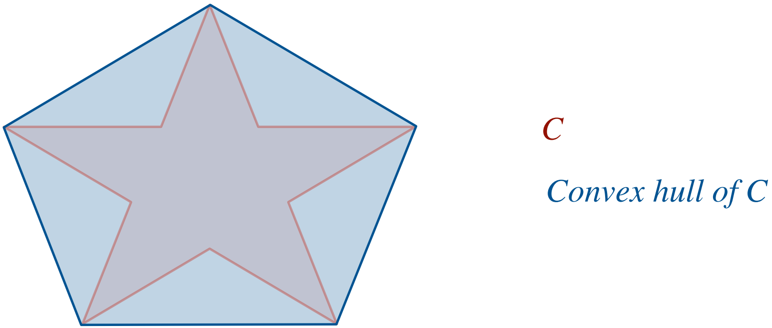


C

Convex hull of C

Definition

Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.

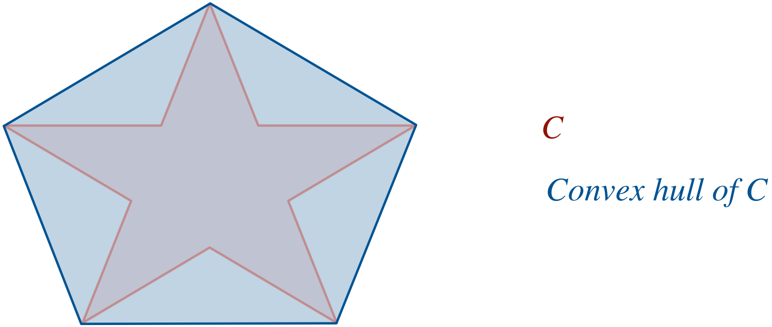


Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C?

Definition

Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.

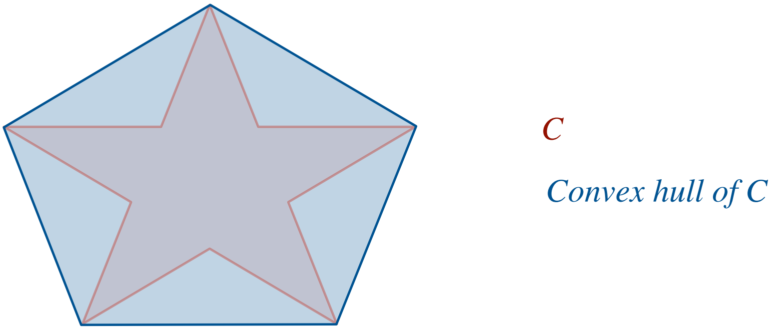


Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

Definition

Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.



Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

⟹ The notion of a convex hull is well defined.

Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

Why?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C

Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

Why?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C

Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$

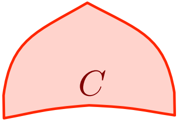
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



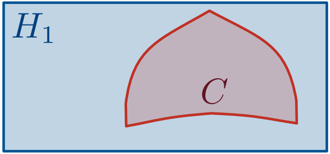
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



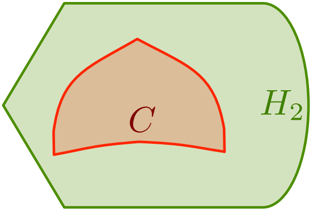
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



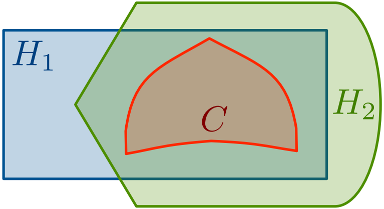
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



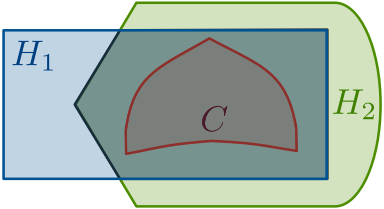
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

Why?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



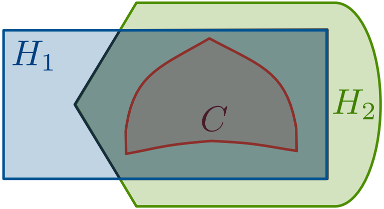
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



⇒

- $C \subseteq H_1 \cap H_2$,

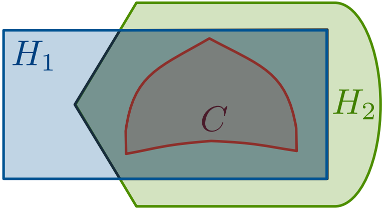
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



$\Rightarrow$

- $C \subseteq H_1 \cap H_2$,
- $H_1 \cap H_2$ is convex

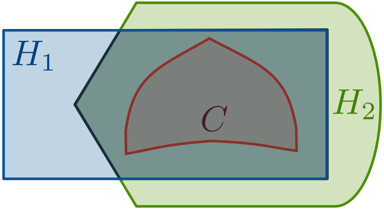
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



$\Rightarrow$

- $C \subseteq H_1 \cap H_2$,
- $H_1 \cap H_2$ is convex

However, $H_1 \cap H_2$ is smaller than both H_1 and H_2. This is a contradiction.

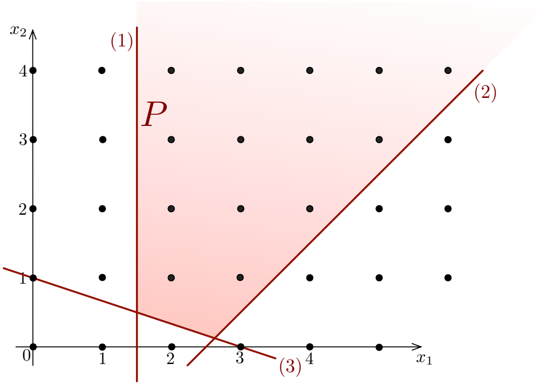
Convex Hulls and Integer Programs

Convex Hulls and Integer Programs

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \, : \, \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$

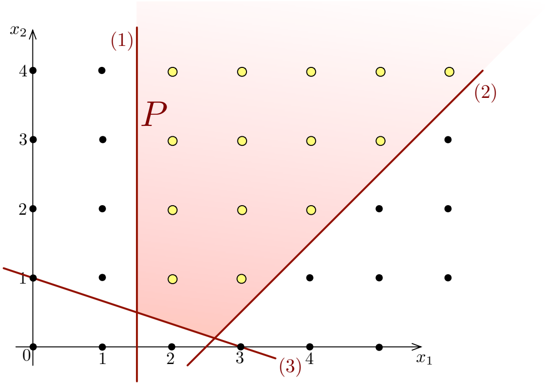
Convex Hulls and Integer Programs

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$



Convex Hulls and Integer Programs

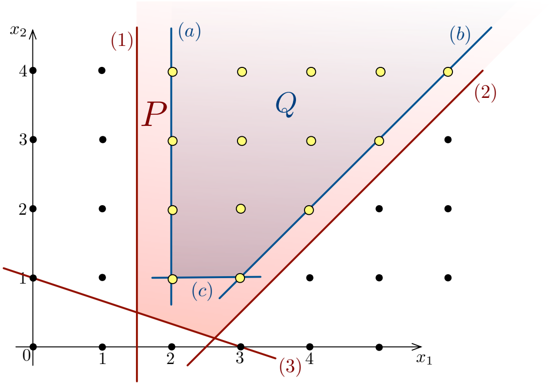
$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;:\; \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$



Integer points in P

Convex Hulls and Integer Programs

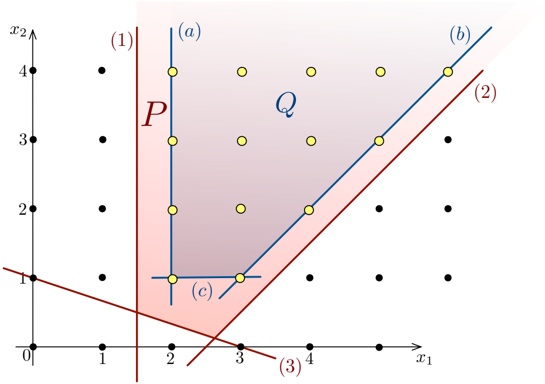
$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$



Q convex hull of integer points in P

Convex Hulls and Integer Programs

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$

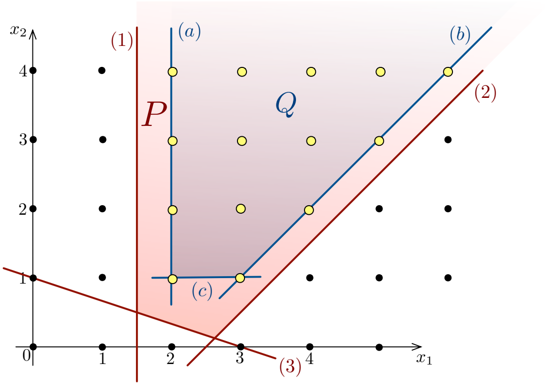


Q convex hull of integer points in P

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \quad \begin{matrix} (a) \\ (b) \\ (c) \end{matrix} \right\}.$$

Convex Hulls and Integer Programs

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$



Q convex hull of integer points in P

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \quad \begin{matrix} (a) \\ (b) \\ (c) \end{matrix} \right\}.$$

Polyhedron

Meyer's Theorem

Consider $P = \{x : Ax \leq b\}$ where A, b are rational.
Then, the convex hull of all integer points in P is a polyhedron.

Meyer's Theorem

Consider $P = \{x : Ax \leq b\}$ where A, b are rational.
Then, the convex hull of all integer points in P is a polyhedron.

(We'll omit the proof)

Meyer's Theorem

Consider $P = \{x : Ax \leq b\}$ where A, b are rational.
Then, the convex hull of all integer points in P is a polyhedron.

(We'll omit the proof)

Remark

The condition that all entries of A and b are rational numbers cannot be excluded from the hypothesis.

Meyer's Theorem

Consider $P = \{x : Ax \le b\}$ where A, b are rational.
Then, the convex hull of all integer points in P is a polyhedron.

(We'll omit the proof)

Remark

The condition that all entries of A and b are rational numbers cannot be excluded from the hypothesis.

Example

Consider

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \le \sqrt{2} x_2,\ x_1, x_2 \ge 1 \right\}.$$

Meyer's Theorem

Consider $P = \{x : Ax \leq b\}$ where A, b are rational.
Then, the convex hull of all integer points in P is a polyhedron.

(We'll omit the proof)

Remark

The condition that all entries of A and b are rational numbers cannot be excluded from the hypothesis.

Example

Consider

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \leq \sqrt{2} x_2,\ x_1, x_2 \geq 1 \right\}.$$

The convex hull of all integer points in P is NOT a polyhedron.

Meyer's Theorem

Consider $P = \{x : Ax \leq b\}$ where A, b are rational.
Then, the convex hull of all integer points in P is a polyhedron.

(We'll omit the proof)

Remark

The condition that all entries of A and b are rational numbers cannot be excluded from the hypothesis.

Example

Consider

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \leq \sqrt{2} x_2,\ x_1, x_2 \geq 1 \right\}.$$

The convex hull of all integer points in P is NOT a polyhedron.

Goal: Use Meyer's theorem to reduce the problem of solving *Integer Programs*, to the problem of solving *Linear Program*.

Let A, b be rational.

$$\max\{c^\top x : Ax \leq b, x \text{ integer}\}. \tag{IP}$$

Let A, b be rational.

$$\max\{c^\top x : Ax \leq b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \leq b'\}$.

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \qquad \text{(IP)}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \qquad \text{(LP)}$$

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \qquad \text{(IP)}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \qquad \text{(LP)}$$

Theorem

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,
- an optimal solution to (IP) is an optimal solution to (LP), and

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,
- an optimal solution to (IP) is an optimal solution to (LP), and
- an extreme optimal solution to (LP) is an optimal solution to (IP).

Let A, b be rational.

$$\max\{c^\top x : Ax \leq b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \leq b'\}$.

$$\max\{c^\top x : A'x \leq b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,
- an optimal solution to (IP) is an optimal solution to (LP), and
- an extreme optimal solution to (LP) is an optimal solution to (IP).

(We'll omit the proofs)

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,
- an optimal solution to (IP) is an optimal solution to (LP), and
- an extreme optimal solution to (LP) is an optimal solution to (IP).

(We'll omit the proofs)

Conceptual way of solving (IP):

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,
- an optimal solution to (IP) is an optimal solution to (LP), and
- an extreme optimal solution to (LP) is an optimal solution to (IP).

(We'll omit the proofs)

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Let A, b be rational.

$$\max\{c^\top x : Ax \leq b, x \text{ integer}\}. \tag{IP}$$

The convex hull of the feasible sol. of (IP) is a polyhedron $\{x : A'x \leq b'\}$.

$$\max\{c^\top x : A'x \leq b'\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,
- an optimal solution to (IP) is an optimal solution to (LP), and
- an extreme optimal solution to (LP) is an optimal solution to (IP).

(We'll omit the proofs)

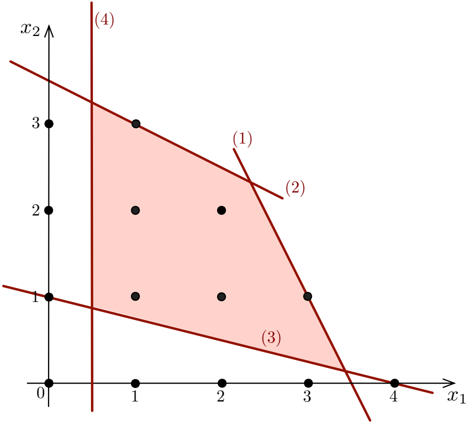
Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution to (LP).

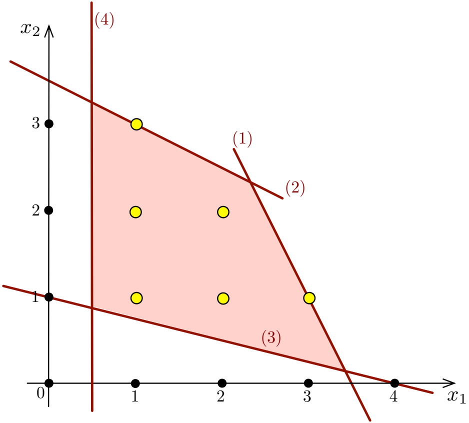
$$\max\left\{ \begin{pmatrix}1 & 1\end{pmatrix} x : \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -4 \\ -1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 7 \\ 7 \\ -4 \\ -\frac{1}{2} \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix} \qquad x \text{ integer} \right\} \qquad \text{(IP)}$$

$$\max\left\{ \begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -4 \\ -1 & 0 \end{pmatrix} x \le \begin{pmatrix} 7 \\ 7 \\ -4 \\ -\frac{1}{2} \end{pmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix} \quad x \text{ integer} \right\} \qquad \text{(IP)}$$



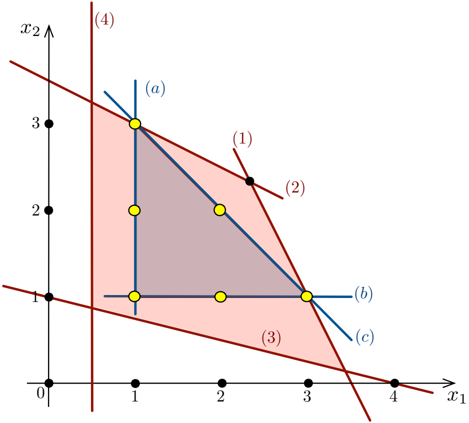
Feasible region of the LP relaxation of (IP)

$$\max\left\{ \begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -4 \\ -1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 7 \\ 7 \\ -4 \\ -\frac{1}{2} \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix} \quad x \text{ integer} \right\} \qquad \text{(IP)}$$



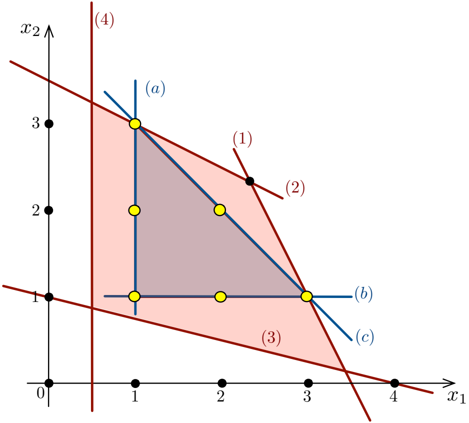
The feasible region of (IP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -4 \\ -1 & 0 \end{pmatrix} x \le \begin{pmatrix} 7 \\ 7 \\ -4 \\ -\frac{1}{2} \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix} \quad x \text{ integer} \right\} \qquad \text{(IP)}$$



The convex hull of the feasible region of (IP)

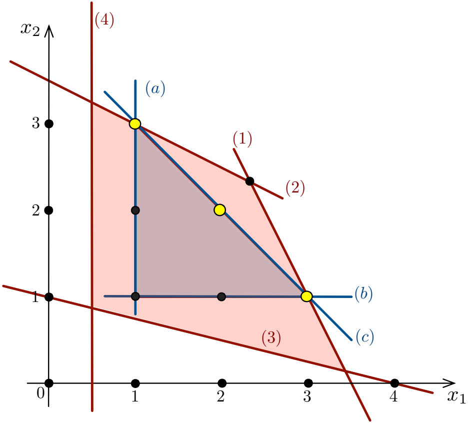
$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}2 & 1\\ 1 & 2\\ -1 & -4\\ -1 & 0\end{pmatrix}x \le \begin{pmatrix}7\\ 7\\ -4\\ -\frac{1}{2}\end{pmatrix}\quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix}\quad x \text{ integer}\right\} \qquad \text{(IP)}$$



The convex hull of the feasible region of (IP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix}x \le \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix}\quad \begin{matrix}(a)\\(b)\\(c)\end{matrix}\right\} \qquad \text{(LP)}$$

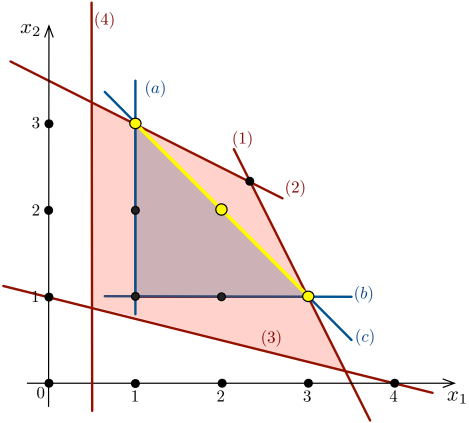
$$\max\left\{ \begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix}2 & 1\\ 1 & 2\\ -1 & -4\\ -1 & 0\end{pmatrix} x \le \begin{pmatrix}7\\ 7\\ -4\\ -\frac{1}{2}\end{pmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix} \quad x \text{ integer} \right\} \qquad \text{(IP)}$$



Optimal solutions of (IP)

$$\max\left\{ \begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix} x \le \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix} \quad \begin{matrix}(a)\\(b)\\(c)\end{matrix} \right\} \qquad \text{(LP)}$$

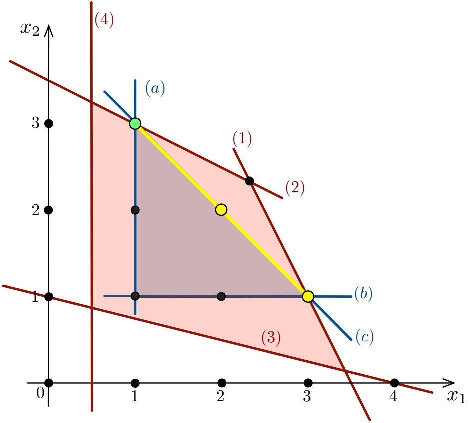
$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}2 & 1\\ 1 & 2\\ -1 & -4\\ -1 & 0\end{pmatrix}x \le \begin{pmatrix}7\\ 7\\ -4\\ -\frac{1}{2}\end{pmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix} \quad x \text{ integer}\right\} \qquad \text{(IP)}$$



Optimal solutions of (LP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix}x \le \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix} \quad \begin{matrix}(a)\\(b)\\(c)\end{matrix}\right\} \qquad \text{(LP)}$$

$$\max \left\{ \begin{pmatrix} 1 & 1 \end{pmatrix} x : \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -4 \\ -1 & 0 \end{pmatrix} x \le \begin{pmatrix} 7 \\ 7 \\ -4 \\ -\frac{1}{2} \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix} \quad x \text{ integer} \right\} \qquad \text{(IP)}$$



Extreme optimal solution of (LP)

$$\max \left\{ \begin{pmatrix} 1 & 1 \end{pmatrix} x : \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} \quad \begin{matrix} (a) \\ (b) \\ (c) \end{matrix} \right\} \qquad \text{(LP)}$$

$$\max\{c^\top x : Ax \le b, x \text{ integer}\} \tag{IP}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\} \tag{IP}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

Remark

This is NOT a practical way to solve an Integer Program.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\} \tag{IP}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \le b'\}$.

$$\max\{c^\top x : A'x \le b'\} \tag{LP}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

Remark

This is NOT a practical way to solve an Integer Program.

WHY NOT?

$$\max\{c^\top x : Ax \leq b, x \text{ integer}\} \tag{IP}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \leq b'\}$.

$$\max\{c^\top x : A'x \leq b'\} \tag{LP}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

Remark

This is NOT a practical way to solve an Integer Program.

WHY NOT?

- We do not know how to compute A', b', and

$$\max\{c^\top x : Ax \leq b, x \text{ integer}\} \tag{IP}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \leq b'\}$.

$$\max\{c^\top x : A'x \leq b'\} \tag{LP}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

Remark

This is NOT a practical way to solve an Integer Program.

WHY NOT?

- We do not know how to compute A', b', and
- A', b' can be MUCH more complicated than A, b.

Question

How do we fix these problems?

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.
- We defined the notion of convex hulls.

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.
- We defined the notion of convex hulls.
- The convex hull of the integer points in a rational polyhedron is a polyhedron.

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.
- We defined the notion of convex hulls.
- The convex hull of the integer points in a rational polyhedron is a polyhedron.
- Integer programming reduces to Linear programming,

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not extend to Integer Programs.
- We defined the notion of convex hulls.
- The convex hull of the integer points in a rational polyhedron is a polyhedron.
- Integer programming reduces to Linear programming,
  but it is NOT a practical reduction.