Module 6: Nonlinear Programs (the KKT theorem)

$$
\begin{array}{llcl}
\min & -x_1 - x_2 & & \\
\text{s.t.} & & & \\
& -x_2 + x_1^2 & \leq 0 & (1) \\
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\end{array}
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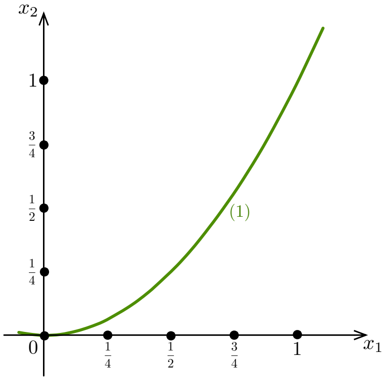

(1) $x_2 \geq x_1^2$;

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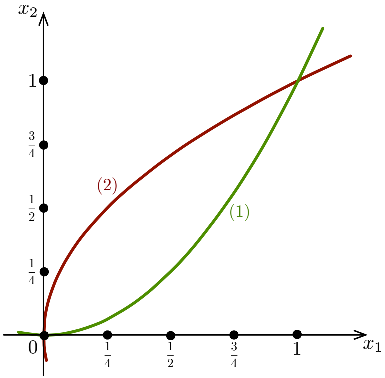
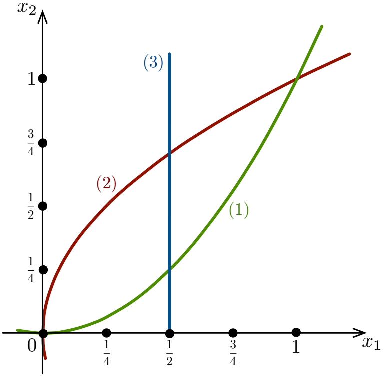


(1) $x_2 \geq x_1^2$;

(2) $x_1 \geq x_2^2$;

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & -x_2 + x_1^2 & \leq & 0 & (1) \\ & -x_1 + x_2^2 & \leq & 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq & 0 & (3) \end{array}$$



(1) $x_2 \geq x_1^2$;

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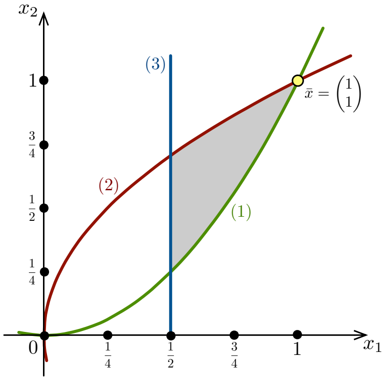

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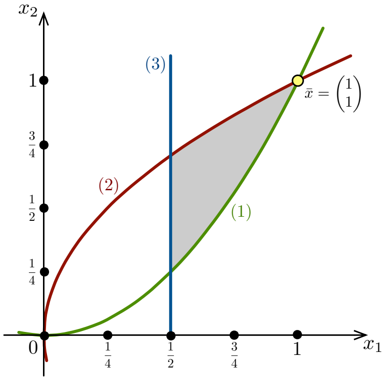

Claim

$\bar{x} = (1,1)^\top$ is an optimal solution to the NLP.

min	$-x_1 - x_2$			
s.t.				
	$-x_2 + x_1^2$	$\leq$	0	(1)
	$-x_1 + x_2^2$	$\leq$	0	(2)
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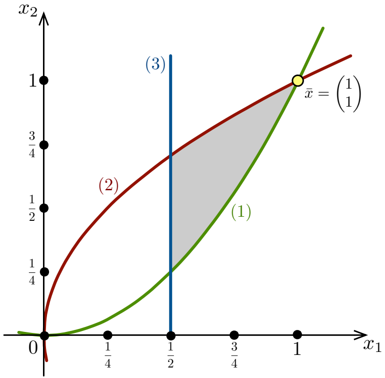

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How do we prove this?

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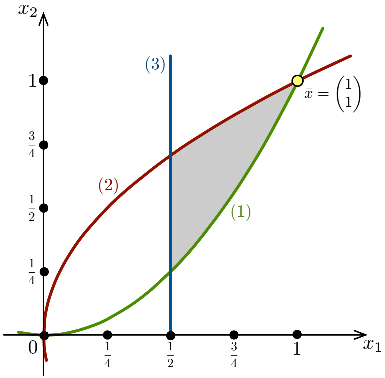

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How do we prove this?

Step 1. Find a relaxation of the NLP.

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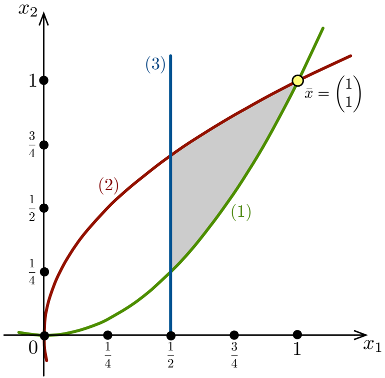

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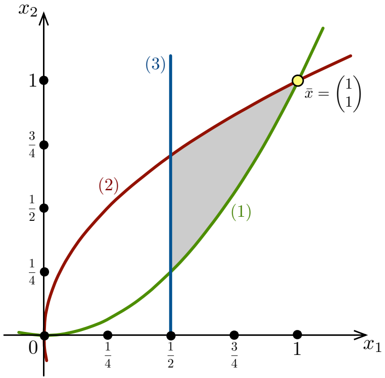

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$\bar{x} = (1,1)^\top$ is an optimal solution to the NLP.

How do we prove this?

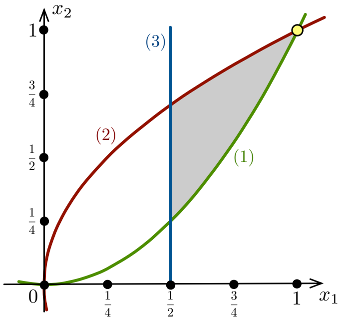
Step 1. Find a relaxation of the NLP.

Step 2. Prove $\bar{x}$ is optimal for the relaxation.

Step 3. Deduce that $\bar{x}$ is optimal for the NLP.

Original NLP

$$\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & -x_2 + x_1^2 & \leq & 0 & (1) \\ & -x_1 + x_2^2 & \leq & 0 & (2) \\ & -x_1 + \frac{1}{2} & \leq & 0 & (3) \end{array}$$



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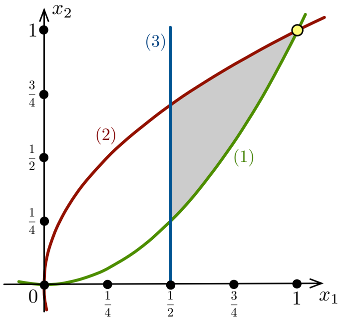

Relaxation

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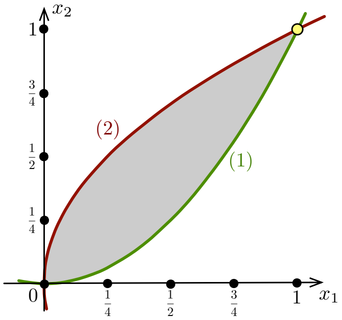

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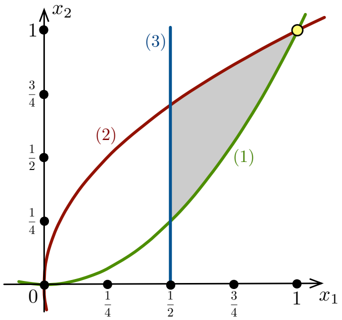
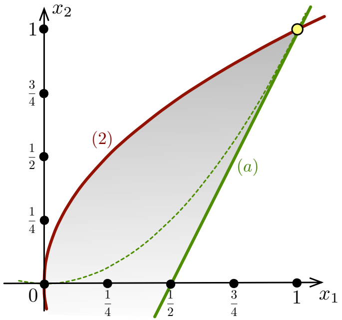


New relaxation

$$\begin{array}{llll} \min & -x_1 - x_2 & & \\ \text{s.t.} & & & \\ & 2x_1 - x_2 & \leq 1 & (a) \\ & -x_1 + x_2^2 & \leq 0 & (2) \end{array}$$



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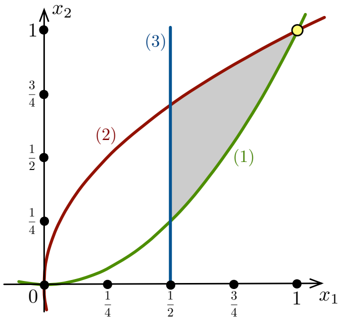

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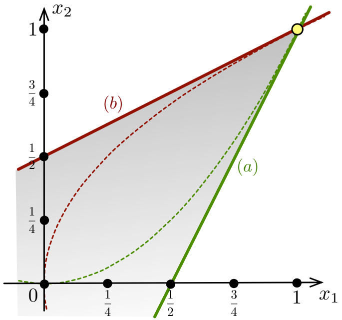

Claim

$\bar{x} = (1, 1)^\top$ is an optimal solution to

$$\boxed{\begin{array}{lrcll} \min & -x_1 - x_2 & & & \\ \text{s.t.} & & & & \\ & 2x_1 - x_2 & \leq & 1 & (a) \\ & -x_1 + 2x_2 & \leq & 1 & (b) \end{array}}$$

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$\bar{x} = (1,1)^\top$ is an optimal solution to

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Proof

Tight constraints for $\bar{x}$ are (a) and (b).

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$$\begin{pmatrix}1\\1\end{pmatrix} \overset{?}{\in} cone\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}$$

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$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \times \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 1 \times \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \checkmark$$

Original NLP

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⇒ $\bar{x}$ is an optimal solution to the *original NLP*

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$\bar{x} = (1,1)^\top$ is an optimal solution to the relaxation

$\Rightarrow$ $\bar{x}$ is an optimal solution to the *original NLP*

Question

Can we do this in general?

Original NLP

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Can we do this in general? YES

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$\bar{x} = (1,1)^\top$ is an optimal solution to the relaxation

$\Rightarrow$ $\bar{x}$ is an optimal solution to the *original NLP*

Question

Can we do this in general? YES

The key tool we'll use is subgradients.

Definition

Let $f : \Re^n \to \Re$ be a convex function

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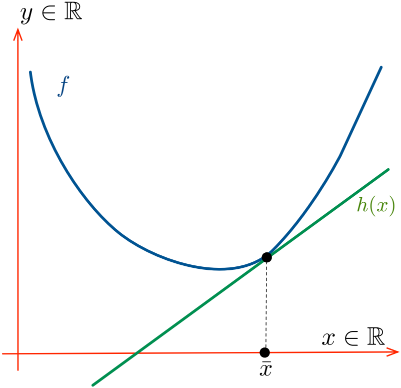

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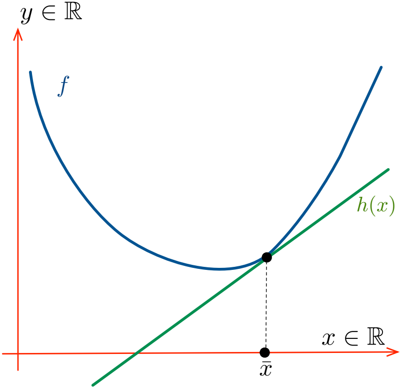
unique subgradient



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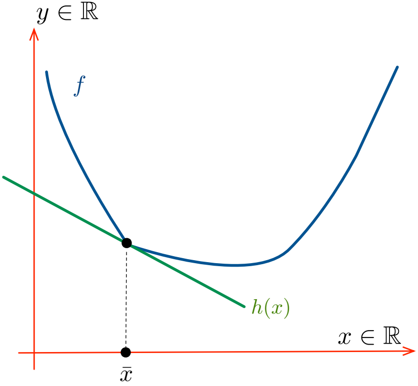

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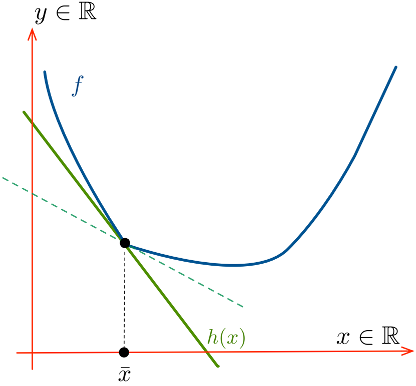

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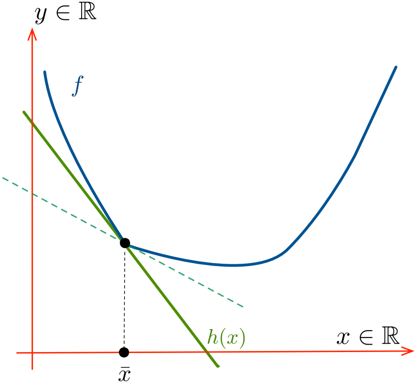
h is a lower bound for f

NOT UNIQUE



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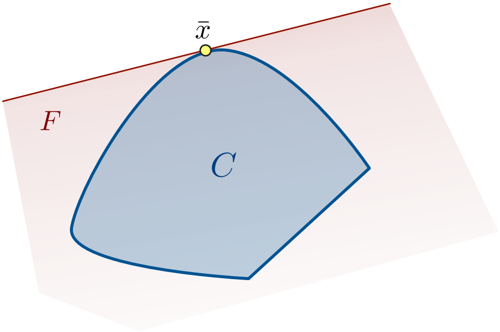

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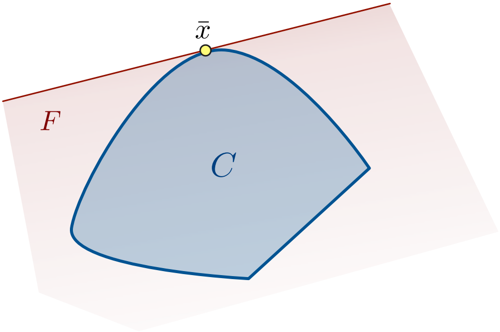

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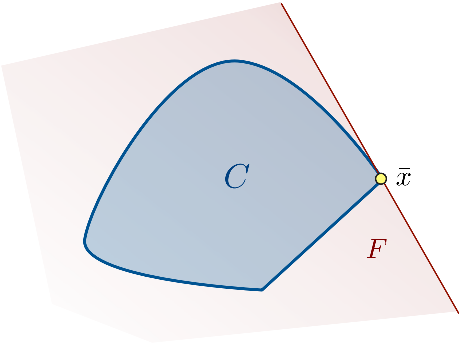

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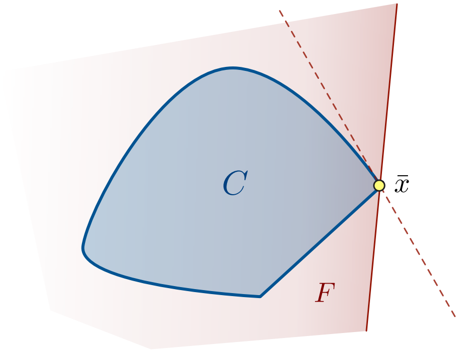

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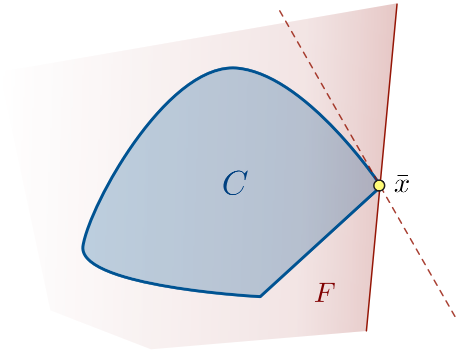

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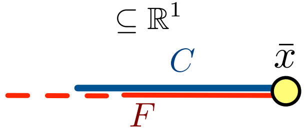

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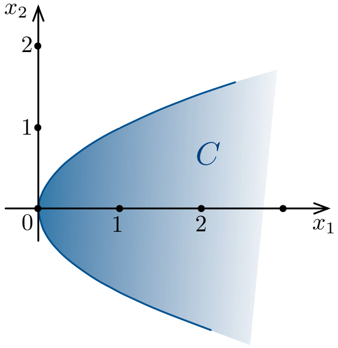

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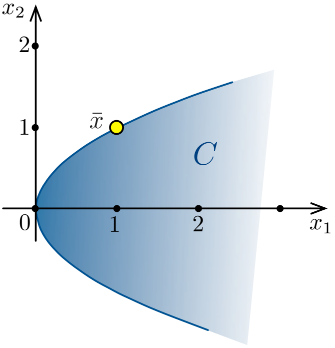

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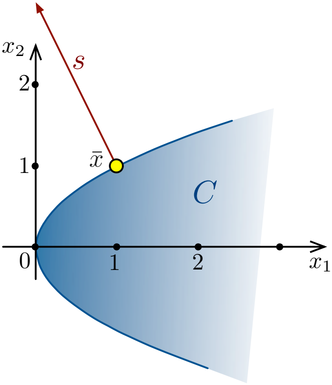

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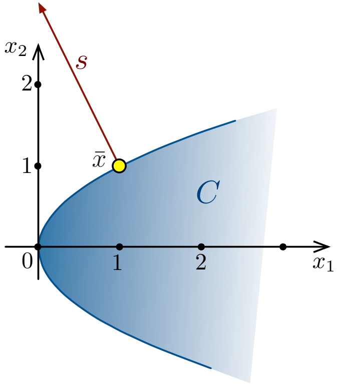

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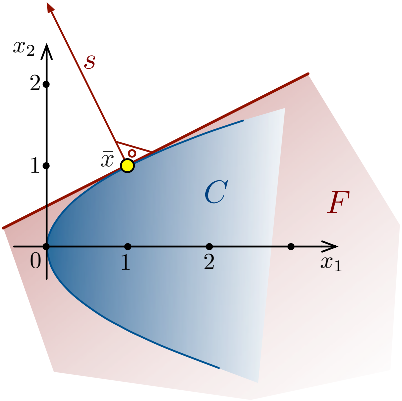

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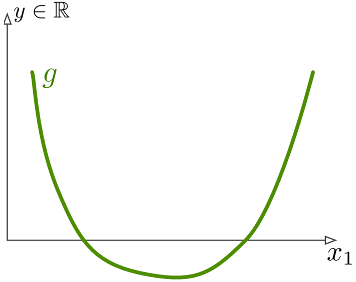

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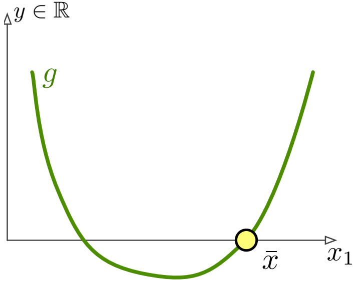

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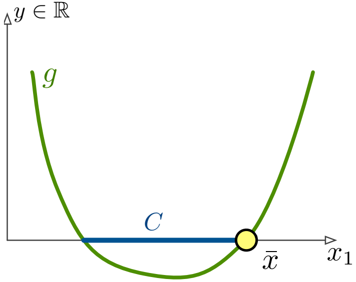

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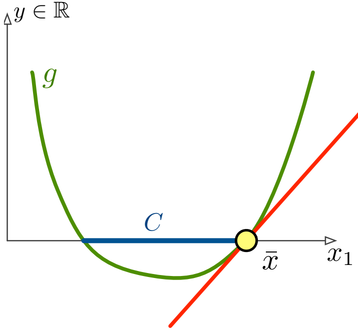

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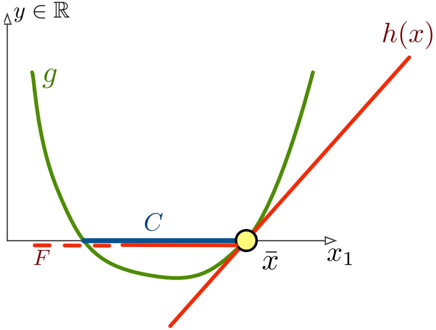

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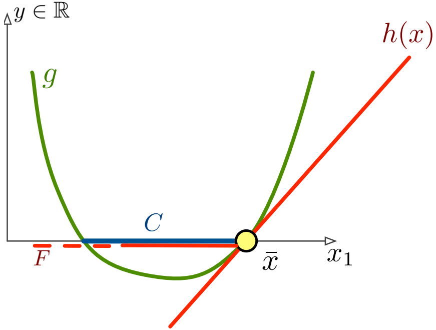

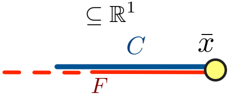

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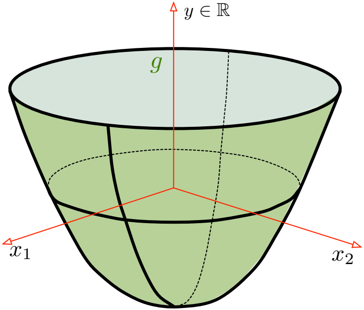

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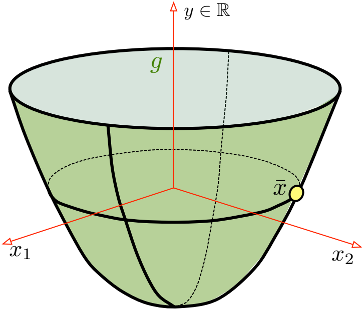

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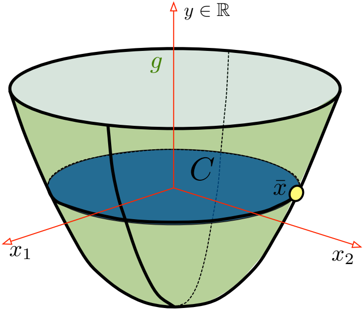

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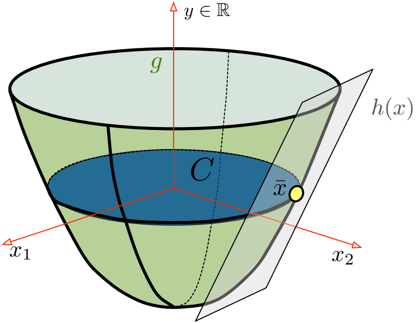

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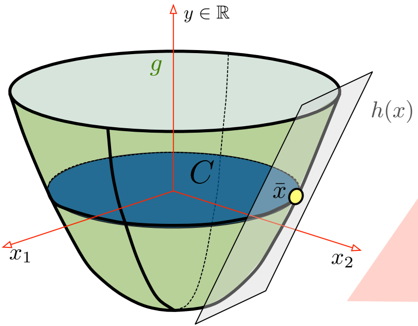

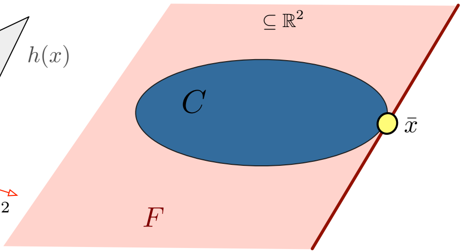

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➡

WE USE IT TO CONSTRUCT RELAXATIONS OF NLPS

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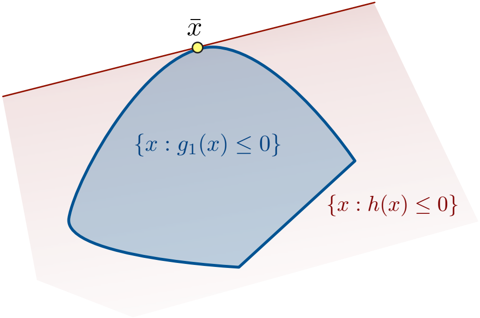

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$$-\begin{pmatrix} -1 \\ -1 \end{pmatrix} \in cone \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} \qquad \Longrightarrow \qquad \bar{x} \text{ optimal.}$$

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We have a relaxation

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$\bar{x}$ is a feasible solution

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If $-c \in cone \left\{ s^{(i)} : i \in I \right\}$ then $\bar{x}$ is optimal.

Proof

We have a relaxation

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We proved that the set of solutions to $g_i(x) \le 0$

is contained in the set of solutions to $g_i(\bar{x}) + s^{(i)}(x - \bar{x}) \le 0$.

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Is there a converse to this result?

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Proposition

Let $f : \Re^n \to \Re$ be function and let $\bar{x} \in \Re^n$.

If the partial derivative $\frac{\partial f(x)}{\partial x_j}$ exists for f at $\bar{x}$ for all $j = 1, \ldots, n$, then the gradient $\nabla f(\bar{x})$ is obtained by evaluating for $\bar{x}$,

$$\left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)^\top .$$

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Compute the gradient of the convex function

$$f(x) = -x_2 + x_1^2$$

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Since $(2, -1)^\top$ is the gradient of f at $\bar{x}$, it is a subgradient as well.

Definition

A feasible solution to $\bar{x}$ is a Slater point of

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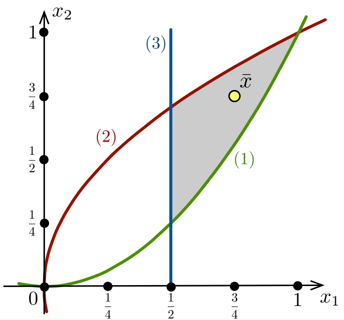

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Remark

We proved the "easy" direction "$\Longleftarrow$".

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- We related subgradients and supporting halfspaces.
- We showed how to relax convex constraints by a linear constraint.
- We gave sufficient conditions for a solution to be optimal.
- We stated the KKT theorem.