Assignment 5

Discussed during the tutorial on Thursday, November 24th, 2022

5.1 (10 points)

(i) Prove that the following LP is infeasible:

$$\begin{aligned}
\max \quad & 3x_1 + 4x_2 + 6x_3 & \\
s.t. \quad & 3x_1 + 5x_2 - 6x_3 &= 4 \\
& x_1 + 3x_2 - 4x_3 &= 2 \\
& -x_1 + x_2 - x_3 &= -1 \\
& x_1, x_2, x_3 &\geq 0 \ .
\end{aligned}$$

(ii) Prove that the following LP is unbounded:

$$\begin{aligned}
\max \ & -x_3 + x_4 & \\
s.t. \ & x_1 + x_3 - x_4 &= 1 \\
& x_2 + 2x_3 - x_4 &= 2 \\
& x_1, x_2, x_3, x_4 &\geq 0 \ .
\end{aligned}$$

(iii) Prove that the following LP $\max\{c^T x \mid Ax = b, x \geq 0\}$, where $A = \begin{pmatrix} 4 & 2 & 1 & -6 & 1 \\ -1 & 1 & -4 & 1 & 3 \\ 3 & -6 & 5 & 3 & -5 \end{pmatrix}$, $b = (11, -2, -8)^T$ and $c = (1, -2, 1, 1, 1)^T$, is unbounded.

(iv) For each of the problems in parts (ii) and (iii), give a feasible solution having objective value exactly 5000.

5.2 (10 points) Convert the following LPs into SEF.

(i) $\min(2, -1, 4, 2, 4)(x_1, x_2, x_3, x_4, x_5)^T$ subject to $Ax \leq b, x \geq 0$, where $b = (1, 2, 3, 4)^T$ and $A = \begin{pmatrix} 1 & 2 & 4 & 7 & 3 \\ 2 & 8 & 9 & 0 & 0 \\ 1 & 1 & 0 & 2 & 6 \\ -3 & 4 & 3 & 1 & -1 \end{pmatrix}$.

(ii) Let A, B, D be matrices and b, c, d, f be vectors, all of suitable dimensions. Convert the following LP with variables x, y (which are vectors) into SEF:

$$\min c^T x + d^T y \text{ subject to } Ax \geq b, Bx + Dy = f, x \geq 0 \ .$$

Note that the variables y are free.

5.3 (10 points) Let A be an $m \times n$ matrix and consider the following LP (P):

$$\max c^T x \text{ subject to } Ax = b, x_j \geq 0 \text{ for } j = 1, \ldots, n-1 \ .$$

Convert (P) into an LP (P') in SEF by replacing the free variable x_n by two variables x_n^+, x_n^-. Show that no basic solution $\bar{x}$ of (P') satisfies $\bar{x}_n^+ > 0$ and $\bar{x}_n^- > 0$.

5.4 (10 points) Consider the LP $\max\{c^T x \mid Ax = b, x \geq 0\}$. Assume that the rows of A are linearly independent. Let $x^{(1)}, x^{(2)}$ be two distinct feasible solutions of (P) and define $\bar{x} = x^{(1)}/2 + x^{(2)}/2$.

(i) Show that $\bar{x}$ is a feasible solution to (P).

(ii) Show that if $x^{(1)}, x^{(2)}$ are optimal solutions to (P), then so is $\bar{x}$.

(iii) Show that $\bar{x}$ is not a basic feasible solution.
Hint: Proceed by contradiction. Suppose that $\bar{x}$ is a basic feasible solution for some basis B. Denote by N the indices of columns of A not in B. Then show that $\bar{x}_N = x_N^{(1)} = x_N^{(2)} = 0$ and that $x_B^{(1)} = x_B^{(2)}$.

(iv) Deduce from (ii) and (iii) that if (P) has two optimal solutions, then (P) has an optimal solution that is not a basic feasible solution.

Upload your solutions as a `.pdf`-file to the course page on the TUHH e-learning portal until 8am on 22nd November.