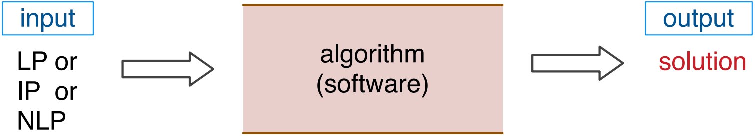
Module 2: Linear programs (Possible outcomes)

What does solving an optimization problem mean?



What does solving an optimization problem mean?

input

LP or
IP or
NLP

algorithm
(software)

output

solution

$$
\begin{array}{ll}
\max & 2x_1 - 3x_2 \\
\text{s.t.} & \\
& x_1 + x_2 \le 1 \\
& x_1, x_2 \ge 0
\end{array}
$$

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⇒ algorithm (software) ⇒

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⇒

$x_1 = 1,$

$x_2 = 0$

Optimal Solution

What does solving an optimization problem mean?

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algorithm
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solution

$$\begin{aligned} \max \quad & 2x_1 - 3x_2 \\ \text{s.t.} \quad & \\ & x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$x_1 = 1, \quad x_2 = 0$$

Optimal Solution

Remark

Sometimes the answer is not so straightforward!!!

Definition

An assignment of values to each of the variables is a feasible solution if all the constraints are satisfied.

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$$\max \quad x_1 - x_2$$

s.t.

$$\begin{array}{rcrcr} 4x_1 & + & x_2 & \geq & 2 \\ 2x_1 & - & 3x_2 & \leq & 4 \end{array}$$

$$x_1,\ x_2 \geq 0$$

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$$x_1 = 1$$

$$x_2 = 3$$

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$x_1 = 1$

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Feasible solution

⇓

Problem is feasible

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$x_1 = 3$

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NOT feasible solution

But problem is feasible.

Definition

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$$\boxed{\begin{array}{ll} \max & x_1 \\ \text{s.t.} & \\ & x_1 \leq 1 \\ & x_2 \geq 1 \end{array}}$$

$x_1 = 1, x_2 = \alpha$ optimal for all $\alpha \geq 1$.

Definition

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$x_1 = 1, x_2 = \alpha$ optimal for all $\alpha \geq 1$.

Remark

An optimization problem can have several optimal solutions.

Question

Does the following linear program have an optimal solution?

$$\max \quad x_1$$

s.t.

$$\begin{aligned} x_1 &\geq 2 \\ x_1 &\leq 1 \end{aligned}$$

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$$\begin{array}{lrcl} \max & x_1 & & \\ \text{s.t.} & & & \\ & x_1 & \geq & 2 \\ & x_1 & \leq & 1 \end{array}$$

Infeasible problem, so no optimal solution

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Does every feasible optimization problem have an optimal solution? NO

$$\begin{array}{ll} \max & x_1 \\ \text{s.t.} & \\ & x_1 \geq 1 \end{array}$$

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Infeasible problem, so
no optimal solution

Question

Does every feasible optimization problem have an optimal solution? NO

$$\boxed{\begin{array}{ll} \max & x_1 \\ \text{s.t.} & \\ & x_1 \geq 1 \end{array}}$$

Feasible ($x_1 = 1$),
but still no optimal solution!!!

Definition

- A maximization problem is unbounded if for every value M there exists a feasible solution with objective value greater than M.

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We have seen three possible outcomes for an optimization problem:

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- It has an optimal solution
- It is infeasible

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Can anything else happen?

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We have seen three possible outcomes for an optimization problem:

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Can anything else happen? YES

Consider,

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

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- Feasible: set $x = 0$.

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Proof

Suppose for a contradiction x is optimal solution.

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Suppose for a contradiction x is optimal solution. Let

$$x' := \frac{x+1}{2}.$$

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- But no optimal solution!

Proof

Suppose for a contradiction x is optimal solution. Let

$$x' := \frac{x+1}{2}.$$

Then $x' < 1$ feasible. Moreover, $x' > x$.

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Any other example without strict inequalities?

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Any other example without strict inequalities? YES

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$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

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Exercise

Check this optimization problem has no optimal solution.

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

Not a linear program
Strict inequality

$$\boxed{\begin{aligned} &\min \quad \frac{1}{x} \\ &\text{s.t.} \\ &\qquad\quad x \geq 1 \end{aligned}}$$

Not a linear program

Objective function non-linear

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Remark

Linear programs are nicer than general optimization problems.

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Not a linear program
Objective function non-linear

Remark

Linear programs are nicer than general optimization problems.

Fundamental theorem of linear programming

For any linear program one of the following holds:

- It has an optimal solution
- It is infeasible
- It is unbounded

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Linear programs are nicer than general optimization problems.

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For any linear program exactly one of the following holds:

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- It has an optimal solution
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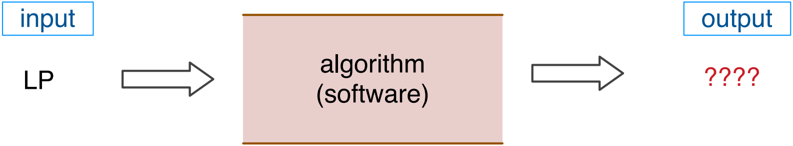
We will prove it later in the course.

We can now describe what we mean by solving a linear program,



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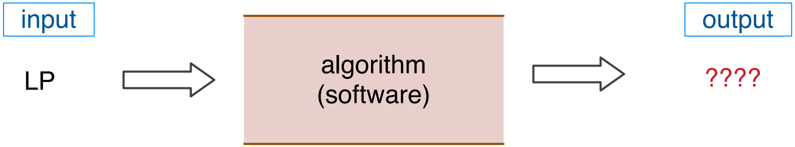

LP has an optimal solution

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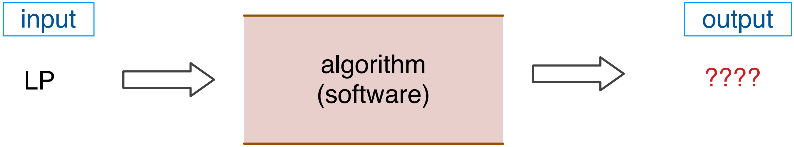

$\boxed{\text{LP has an optimal solution}}$

Return an optimal solution $\bar{x}$

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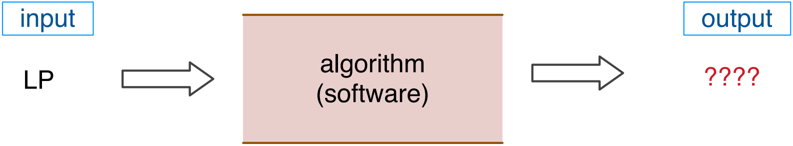

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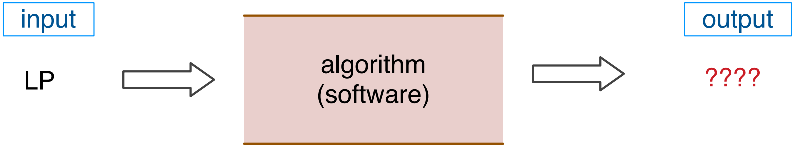

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Say the LP is infeasible

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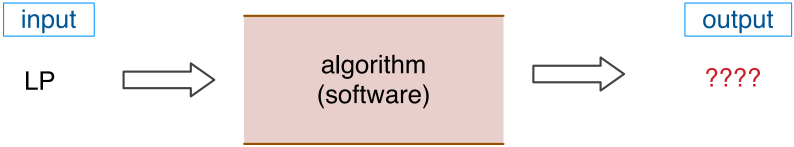

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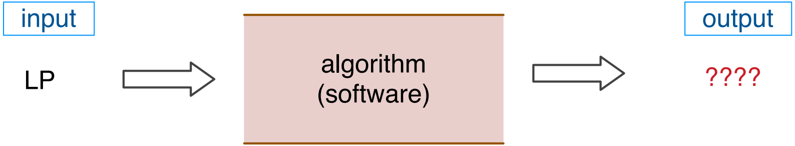

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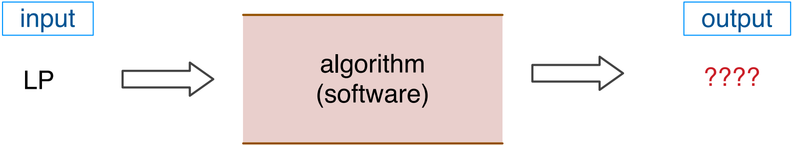

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Return an optimal solution $\bar{x}$

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Say the LP is infeasible

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Remark

Algorithms should justify their answers !!!

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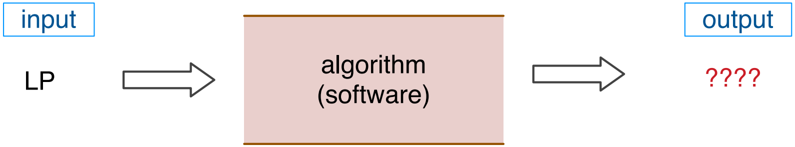

LP has an optimal solution

Return an optimal solution $\bar{x}$ + proof that $\bar{x}$ is optimal.

LP is infeasible.

Return a proof the LP is infeasible.

LP is unbounded.

Return a proof the LP is unbounded.

Remark

Algorithms always need to justify their answers !!!

Recap

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1. Optimization problems can be:

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   (A) infeasible,

Recap

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Recap

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4. By solving an LP we mean
   - indicating which of (A), (B), (C) holds,
   - if (C) holds give an optimal solution,
   - give a proof the answer is correct.