Module 2: Linear Programs (Simplex – A First Attempt)

A Naive Strategy for Solving an LP

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Many details missing!

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Questions

- How do we find a feasible solution?

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- How do we find a feasible solution?
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The SIMPLEX algorithm works along these lines.

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- How do we find a "better" solution?
- Will this ever terminate?

The SIMPLEX algorithm works along these lines.

In this lecture: A first attempt at this algorithm.

A First Example

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Consider

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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\end{array}
$$

Remarks

- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.

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- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.
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The feasible solution has objective value: $4 \times 0 + 3 \times 0 + 7 = 7$.

- Can we find a feasible solution with value larger than 7?

A First Example

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Question

The feasible solution has objective value: $4 \times 0 + 3 \times 0 + 7 = 7$.

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YES!

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Idea

Increase x_1 as much as possible, and keep x_2 unchanged, i.e.,

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Remarks

- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.
- The objective function is $z = 4x_1 + 3x_2 + 7$.

Idea

Increase x_1 as much as possible, and keep x_2 unchanged, i.e.,

$$\begin{aligned} x_1 &= t \qquad \text{for some } t \geq 0 \text{ as large as possible} \\ x_2 &= 0 \end{aligned}$$

$$
\boxed{\begin{array}{ll}
\max & (4,3,0,0)x + 7 \\
\text{s.t.} & \\
& \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
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\end{array}}
$$

$$
\begin{aligned}
x_1 &= t \\
x_2 &= 0 \\
x_3 &= ? \\
x_4 &= ?
\end{aligned}
$$

Choose $t \geq 0$ as large as possible.

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Choose $t \geq 0$ as large as possible.

It needs to satisfy

1. the equality constraints, and

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Choose $t \geq 0$ as large as possible.

It needs to satisfy

1. the equality constraints, and
2. the non-negativity constraints.

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$

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$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x$$

Satisfying the Equality Constraints

$$\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$\begin{aligned} x_1 &= t \\ x_2 &= 0 \\ x_3 &= ? \\ x_4 &= ? \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

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\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\
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\end{aligned}$$

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$$\boxed{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

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$$\boxed{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

Remark

Equality constraints hold for any choice of t.

Satisfying the Non-Negativity Constraints

$$\boxed{\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

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Choose $t \geq 0$ as large as possible.

Satisfying the Non-Negativity Constraints

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Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

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$x_3 = 2 - 3t \geq 0 \Longrightarrow t \leq \frac{2}{3}$

Satisfying the Non-Negativity Constraints

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Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

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$x_3 = 2 - 3t \geq 0 \Rightarrow t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0 \Rightarrow t \leq 1$

Thus, the largest possible t is $\min\left\{1, \frac{2}{3}\right\} = \frac{2}{3}$.

Satisfying the Non-Negativity Constraints

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Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0$ ➡ $t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0$ ➡ $t \leq 1$

Thus, the largest possible t is $\min\left\{1, \frac{2}{3}\right\} = \frac{2}{3}$. The new solution is

$$x = (t,\ 0,\ 2-3t,\ 1-t)^\top = \left(\frac{2}{3}, 0, 0, \frac{1}{3}\right)^\top$$

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

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$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$

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Question

Is the new solution optimal?

Repeating the Argument?

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Question

Is the new solution optimal? NO!

Repeating the Argument?

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s.t.

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$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$
$$x_2 = 0$$
$$x_3 = 0$$
$$x_4 = \frac{1}{3}$$

Question

Is the new solution optimal? NO!

Question

Can we use the same trick to get a better solution?

Repeating the Argument?

$$
\begin{aligned}
\max \quad & (4,3,0,0)x + 7 \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \frac{2}{3} \\
x_2 &= 0 \\
x_3 &= 0 \\
x_4 &= \frac{1}{3}
\end{aligned}
$$

Question

Is the new solution optimal? NO!

Question

Can we use the same trick to get a better solution? NO!

Repeating the Argument?

$$\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = \frac{2}{3}$$
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Is the new solution optimal? NO!

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Can we use the same trick to get a better solution? NO!

What made it work the first time around?

Remark

The LP needs to be in “canonical” form.

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$$
\begin{array}{ll}
\max & \begin{pmatrix}4 & 3 & 0 & 0\end{pmatrix} x + 7 \\
\text{s.t.} & \\
& \begin{pmatrix}3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & \begin{pmatrix} 4 & 3 & 0 & 0 \end{pmatrix} x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$x_1 = 0$

$x_2 = 0$

$x_3 = 2$

$x_4 = 1$

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Revised strategy:

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Step 2. Rewrite LP so that it is in "canonical" form.

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Step 1. Find a feasible solution, x.

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Step 3. If x is optimal, STOP.

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Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Step 3. If x is optimal, STOP.

Step 4. If LP is unbounded, STOP.

Step 5. Find a "better" feasible solution.

Remark

The LP needs to be in "canonical" form.

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& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

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Revised strategy:

Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Step 3. If x is optimal, STOP.

Step 4. If LP is unbounded, STOP.

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From Here to a Complete Algorithm

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First on "To do list":

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First on "To do list":

- Define basis and basic solutions.
- Define canonical forms.