Module 2: Linear Programs (Finding a Feasible Solution)

The Problem

Consider

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}.$$

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To run Simplex, we need a feasible basis.

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Question

How do we find a feasible basis?

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How do we find a feasible basis?

An easier question,

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To run Simplex, we need a feasible basis.

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How do we find a feasible basis?

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How do we find a feasible solution?

These two questions are equivalent.

The Problem

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$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}.$$

To run Simplex, we need a feasible basis.

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How do we find a feasible basis?

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How do we find a feasible solution?

These two questions are equivalent.

Exercise

There is an algorithm that, given a feasible solution, finds a feasible basis.

The Problem

Consider

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}.$$

To run Simplex, we need a feasible basis.

Question

How do we find a feasible basis?

An easier question,

Question

How do we find a feasible solution?

These two questions are equivalent.

Exercise

There is an algorithm that, given a feasible solution, finds a feasible basis.

$\implies$ We will focus on the second question.

The Key Idea

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$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}$$

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Algorithm 1

INPUT: A, b, c, and a feasible solution

OUTPUT: Optimal solution/detect LP unbounded.

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<u>INPUT:</u> A, b, c, and a feasible solution

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Simplex +
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Algorithm 1

<u>INPUT:</u> A, b, c, and a feasible solution

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Algorithm 2

<u>INPUT:</u> A, b, c.

<u>OUTPUT:</u> Feasible solution/detect there is none.

The Key Idea

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Algorithm 1

<u>Input:</u> A, b, c, and a feasible solution

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<u>Input:</u> A, b, c.

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HOW?

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$\underline{\text{Input:}}$ A, b, c, and a feasible solution

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Algorithm 2

$\underline{\text{Input:}}$ A, b, c.

$\underline{\text{Output:}}$ Feasible solution/detect there is none.

HOW?

We will show that...

The Key Idea

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}$$

Algorithm 1

<u>Input:</u> A, b, c, and a feasible solution

<u>Output:</u> Optimal solution/detect LP unbounded.

OK
Simplex +
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Algorithm 2

<u>Input:</u> A, b, c.

<u>Output:</u> Feasible solution/detect there is none.

HOW?

We will show that...

Proposition

We can use Algorithm 1 to get Algorithm 2.

A First Example

A First Example

Problem: Find a feasible solution/detect none exist for

$$
\begin{array}{ll}
\max & (1, 2, -1, 3)x \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

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\begin{aligned}
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& x \geq \mathbf{0}
\end{aligned}
$$

Remark

It does not depend on the objective function.

<u>Problem:</u> Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

<u>Problem:</u> Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.

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Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

min $x_5 + x_6$

s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$x \geq \mathbf{0}$

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x_5, x_6 are the

auxiliary variables

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The auxiliary problem is

<u>Problem:</u> Find a feasible solution/detect none exist for

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$$
\begin{aligned}
&\min \quad x_5 + x_6 \\
&\text{s.t.} \\
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&\qquad x \geq \mathbf{0}
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$$

x_5, x_6 are the
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Remark

The auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and

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x_5, x_6 are the
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Remark

The auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and
- bounded, as 0 is the lower bound.

Problem: Find a feasible solution/detect none exist for

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Remark

The auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and
- bounded, as 0 is the lower bound.

$\Longrightarrow$ Therefore, the auxiliary problem has an optimal solution.

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$x \geq \mathbf{0}$

x_5, x_6 are the
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Step 3. Solve the auxiliary problem using Algorithm 1.

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Step 1. Multiply the equations such that the RHS is non-negative.   OK

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$$\min \quad x_5 + x_6$$
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$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

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$$\begin{array}{ll} \min & x_5 + x_6 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

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$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

since $x_5 = x_6 = 0$.

➡   Therefore, $(2, 1, 0, 0)^\top$ is a feasible solution to $(\star)$.

A Second Example

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$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

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x_4, x_5 are the auxiliary variables

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Step 3. Solve the auxiliary problem using Algorithm 1.

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Step 1. Multiply the equations such that the RHS is non-negative. OK

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$$\min \quad z = x_4 + x_5$$
s.t.
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Step 3. Solve the auxiliary problem using Algorithm 1.

$(0, 0, 1, 0, 3)^\top$ is an optimal solution to the auxiliary problem.

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned}
\min \quad & z = x_4 + x_5 \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}$$

x_4, x_5 are the
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Step 3. Solve the auxiliary problem using Algorithm 1.

$(0, 0, 1, 0, 3)^\top$ is an optimal solution to the auxiliary problem.

However, $(0, 0, 1)^\top$ is NOT a solution to $(\star)$.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{aligned}
\min \quad & z = x_4 + x_5 \\
\text{s.t.} \quad & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{array}{ll}
\min & z = x_4 + x_5 \\
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\end{array}$$

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optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\boxed{\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{array}{ll} \min & z = x_4 + x_5 \\ \text{s.t.} \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x_1', x_2', x_3'.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\boxed{\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x_1', x_2', x_3'.

Then, $(x_1', x_2', x_3', 0, 0)^\top$ is a feasible solution to the auxiliary problem,

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\boxed{\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x_1', x_2', x_3'.

Then, $(x_1', x_2', x_3', 0, 0)^\top$ is a feasible solution to the auxiliary problem,

but that solution has of value 0. This is a contradiction.

Formalize

<u>Problem:</u> Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

$$
\begin{aligned}
\min \quad & z = x_{n+1} + \ldots + x_{n+m} \\
\text{s.t.} \quad & \\
& (\quad A \quad | \quad I \quad)\, x = b \\
& x \geq \mathbf{0}
\end{aligned}
$$

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

$$
\begin{aligned}
\min \quad & z = x_{n+1} + \ldots + x_{n+m} \\
\text{s.t.} \quad & \\
& \left(\quad A \quad | \quad I \quad \right) x = b \\
& x \geq \mathbf{0}
\end{aligned}
$$

$x_{n+1}, \ldots, x_{n+m}$ are the

auxiliary variables

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Step 3. Solve the auxiliary problem using Algorithm 1.

Formalize

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\text{s.t.} \\
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\qquad x \geq \mathbf{0}
\end{array}
$$

$x_{n+1}, \ldots, x_{n+m}$ are the

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Step 3. Solve the auxiliary problem using Algorithm 1.

$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$ is an optimal solution to the auxiliary problem.

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Proposition

If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to $(\star)$.

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$x_{n+1}, \ldots, x_{n+m}$ are the

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Proposition

If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to $(\star)$.

Proof

When $z = 0$, we must have $x_{n+1} = \ldots = x_{n+m} = 0$.

Problem: Find a feasible solution/detect none exist for

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\text{s.t.} \quad & \\
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& x \geq \mathbf{0}
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Proposition

When $z > 0$, then $(\star)$ has no solution.

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \tag{$\star$}$$

$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \left(\begin{array}{c|c} A & I \end{array} \right) x = b \\ & x \geq \mathbf{0} \end{array}$$

the auxiliary problem

optimal solution
$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$

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Proof

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

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\begin{array}{ll}
\min & z = x_{n+1} + \ldots + x_{n+m} \\
\text{s.t.} & \\
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$$

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Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{aligned} \min \quad & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} \quad & \\ & \begin{pmatrix} A & | & I \end{pmatrix} x = b \\ & x \geq \mathbf{0} \end{aligned}$$

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Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$
\begin{array}{ll}
\min & z = x_{n+1} + \ldots + x_{n+m} \\
\text{s.t.} & \\
& \left(\begin{array}{c|c} A & I \end{array} \right) x = b \\
& x \geq \mathbf{0}
\end{array}
$$

the auxiliary problem

optimal solution
$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$

Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

but that solution has of value 0.

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{aligned} \min \quad & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} \quad & \\ & \left(\begin{array}{c|c} A & I \end{array} \right) x = b \\ & x \geq \mathbf{0} \end{aligned}$$

the auxiliary problem

optimal solution
$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$

Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x_1', \ldots, x_n'$.

Then $(x_1', \ldots, x_n', 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

but that solution has of value 0. This is a contradiction.

The 2-Phase Method

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To solve
$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\},$$

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Phase 2. Given that feasible solution:
find an optimal solution/detect LP unbounded.

The 2-Phase Method

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Phase 1. Find a feasible solution/detect none exist.

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find an optimal solution/detect LP unbounded.

Example

Solve the following LP,

$$
\begin{array}{ll}
\max & (1,1,1)x \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

---

min $z = x_4 + x_5$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$x \geq \mathbf{0}$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned}
\min \quad & z = x_4 + x_5 \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}$$

NOT in SEF

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

---

max $z = -x_4 - x_5$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$x \geq \mathbf{0}$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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---

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In SEF

Phase 1. Find a feasible solution/detect none exist for

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$x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

max $z = -x_4 - x_5$
s.t.
$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$
$x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

<div style="border:1px solid">

max $z = -x_4 - x_5$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$x \geq \mathbf{0}$

</div>

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

To rewrite $B = \{4, 5\}$ in canonical form, you can

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{l} \max \quad z = -x_4 - x_5 \\ \text{s.t.} \\ \qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

To rewrite $B = \{4, 5\}$ in canonical form, you can

- use the formulae, OR

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

max $z = -x_4 - x_5$
s.t. $$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$ $x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

To rewrite $B = \{4, 5\}$ in canonical form, you can

- use the formulae, OR
- notice $A_B = I$ and rewrite the objective function as follows...

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

max $z = -x_4 - x_5$
s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$z \;=\; (0 \quad 0 \quad 0 \quad -1 \quad -1) x$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

<table>
<tr><td>

max $\quad z = -x_4 - x_5$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$\qquad x \geq \mathbf{0}$

</td></tr>
</table>

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{aligned} z &= (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\ 0 &= (1 \quad 2 \quad -1 \quad 1 \quad 0)x \quad - \quad 4 \end{aligned}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

max $\quad z = -x_4 - x_5$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$
\begin{aligned}
z &= (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\
0 &= (1 \quad 2 \quad -1 \quad 1 \quad 0)x \quad - \quad 4 \\
0 &= (1 \quad -1 \quad 1 \quad 0 \quad 1)x \quad - \quad 4
\end{aligned}
$$

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$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Step 2. Construct the auxiliary problem.

max $z = -x_4 - x_5$
s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{array}{rcl}
z & = & (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\
0 & = & (1 \quad 2 \quad -1 \quad 1 \quad 0)x \quad - \quad 4 \\
0 & = & (1 \quad -1 \quad 1 \quad 0 \quad 1)x \quad - \quad 4 \\
\hline
z & = & (2 \quad 1 \quad 0 \quad 0 \quad 0)x \quad - \quad 8
\end{array}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{ll} \max & z = (2 \;\; 1 \;\; 0 \;\; 0 \;\; 0) - 8 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

In SEF

feasible basis $B = \{4, 5\}$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{ll} \max & z = (2 \ \ 1 \ \ 0 \ \ 0 \ \ 0) - 8 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

max $z = (2 \;\; 1 \;\; 0 \;\; 0 \;\; 0) - 8$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{ll}
\max & z = (2\ \ 1\ \ 0\ \ 0\ \ 0) - 8 \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

$B = \{1, 4\}$ is an optimal basis with the basic solution $(4, 0, 0, 0, 0)^{\top}$.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.   OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{l}
\max \quad z = (2 \ \ 1 \ \ 0 \ \ 0 \ \ 0) - 8 \\
\text{s.t.} \\
\qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

$B = \{1, 4\}$ is an optimal basis with the basic solution $(4, 0, 0, 0, 0)^\top$.

$z = 0$ implies that $(4, 0, 0)^\top$ is a feasible solution for $(\star)$.

Phase 1. $(4, 0, 0)^\top$ is a feasible solution for

$$
\begin{aligned}
\max \quad & (2, -1, 2)x \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
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$(4, 0, 0)^\top$ is a basic solution.

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Exercise

Show that this will always be the case!

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Thus, for some $i \in \{2, 3\}$, columns 1 and i of A are independent.

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Phase 2. Find an optimal solution/detect LP unbounded.

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$\longrightarrow$

$x = (0, 8, 12)^\top$ is an optimal solution.

Theorem

$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}$$

Consequences

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Run 2-Phase method with Simplex using Bland's rule.

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(Recall that Bland's rule ensures that Simplex terminates.)

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A finite number of basis implies a finite number of basic solutions.

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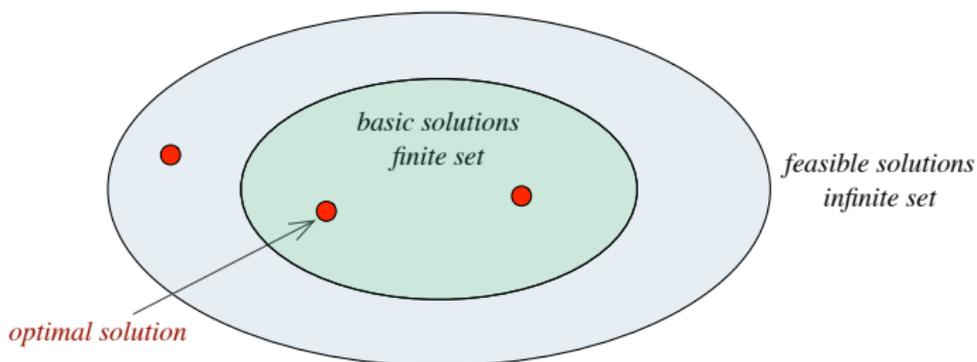$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}$$

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Convert the LP into an equivalent LP in SEF.

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Apply the previous theorem.

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gamegfx/iStock/Thinkstock

Our implementation