Module 3: Duality through examples (Correctness Shortest Path Algorithm)

Recap: Shortest Path Algorithm

Previous lecture: we showed an algorithm for the shortest path problem that computes

- An s,t-path P

Shortest path LP:

$$\begin{aligned} \min \quad & \sum (c_e x_e \,:\, e \in E) \\ \text{s.t.} \quad & \sum (x_e \,:\, e \in \delta(S)) \geq 1 \\ & \qquad\qquad (\delta(S)\ s,t\text{-cut}) \\ & x \geq \mathbb{0} \end{aligned}$$

Shortest path dual:

$$\begin{aligned} \max \quad & \sum (y_S \,:\, \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \leq c_e \\ & \qquad\qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

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Important: $c^T x = \mathbb{1}^T y$

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We will start this lecture with another example!

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Recall the algorithm we developed previously:

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

$y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$

while $t \notin U$ **do**

  Let ab be an edge in $\delta(U)$ of smallest slack for y where $a \in U$, $b \notin U$

  $y_U := \text{slack}_y(ab)$

  $U := U \cup \{b\}$

  change edge ab into an arc $\overrightarrow{ab}$

end while

return A directed st-path P.

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while t ∉ U do
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    y_U := slack_y(ab)
    U := U ∪ {b}
    change edge ab into an arc →ab
end while
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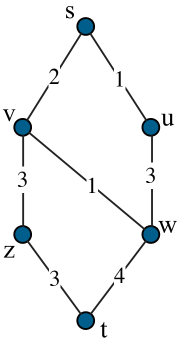

$\longrightarrow$ Run this on the example instance on the right.

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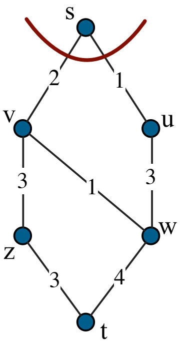

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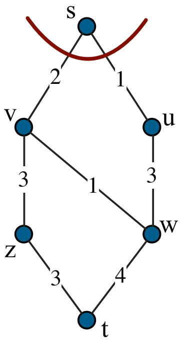
Step 1 $\quad su$ edge with smallest slack in $\delta(U)$



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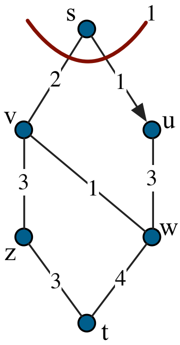
$\longrightarrow$ increase y_U by 1



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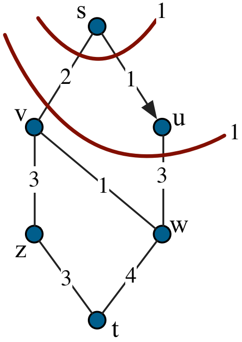

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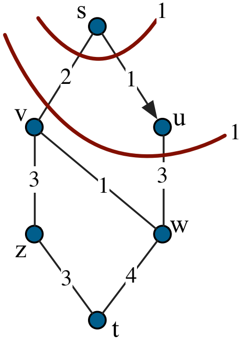

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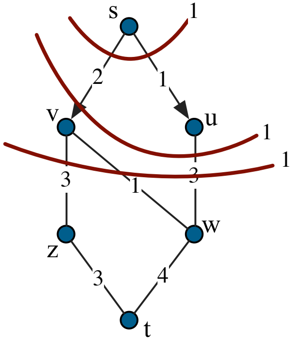

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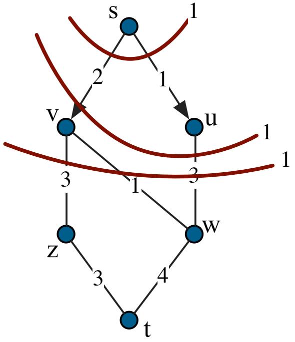

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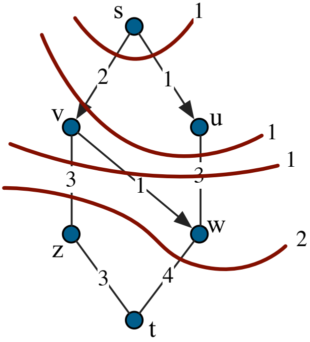

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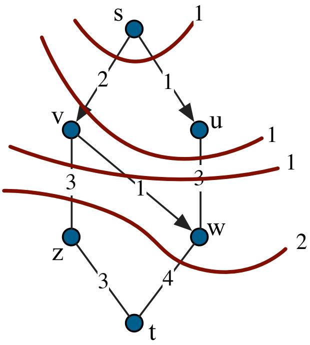
$\longrightarrow$ increase y_U by 2



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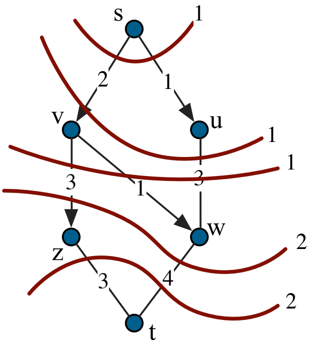
Slack minimizer is wt



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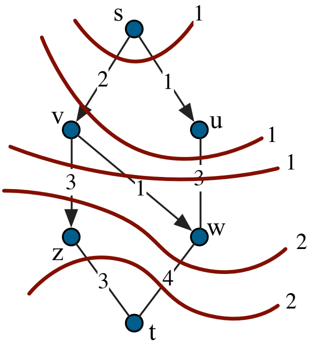

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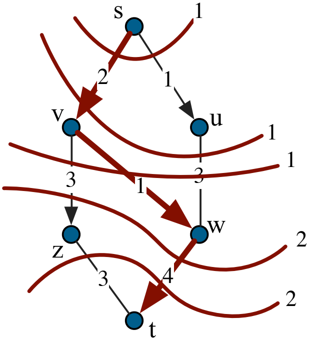

Now: We have a directed s, t-path P of length 7,

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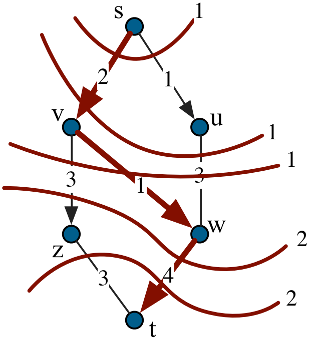

Now: We have a directed s, t-path P of length 7, and a dual feasible solution of the same value!

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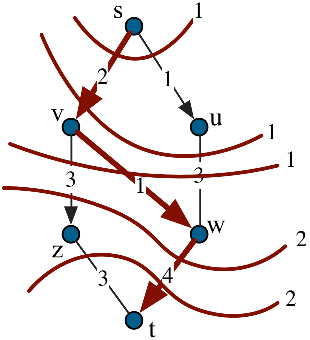
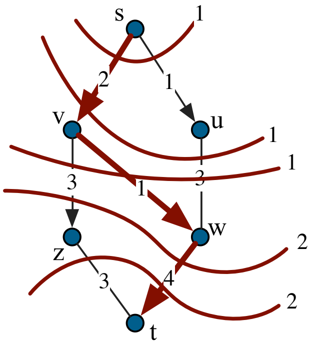


Now: We have a directed s,t-path P of length 7, and a dual feasible solution of the same value!

$\longrightarrow$ P is a shortest path!

Question

Will the algorithm always terminate?



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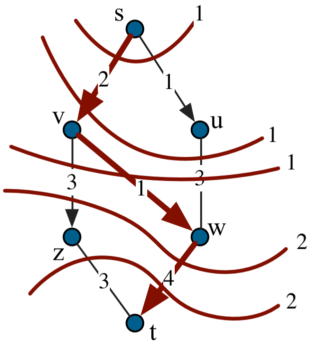
Will the algorithm always terminate? Will it always find an s, t-path P whose length is equal to the value of a feasible dual solution?



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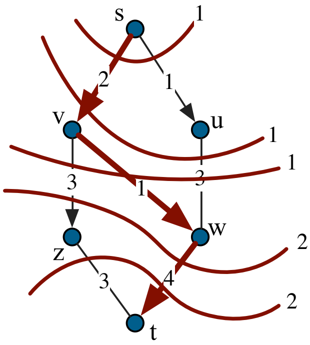
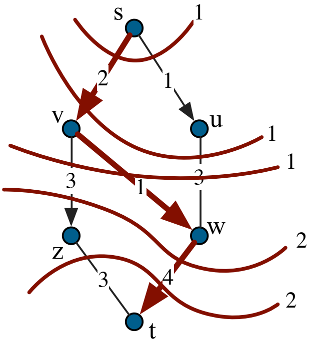


This lecture: We will show the answers to the above are yes & yes!

Recall: the slack of an edge $uv \in E$ for a feasible dual solution y is

$$c_{uv} - \sum(y_U \,:\, e \in \delta(U))$$

.



Revisited: Shortest Path Optimality Conditions

Recall: the slack of an edge $uv \in E$ for a feasible dual solution y is

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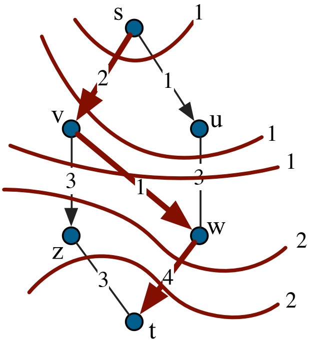
We call an edge $uv \in E$ an equality edge if its slack is 0.



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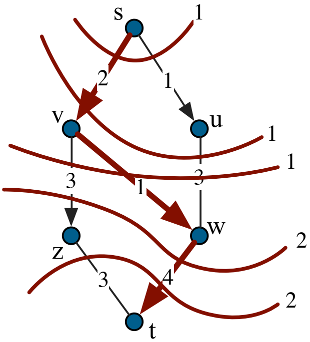
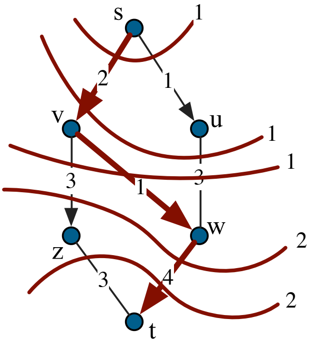
Example: edge vz is an equality edge, and zt is not!



Revisited: Shortest Path Optimality Conditions

We will also call a cut $\delta(U)$ active for a dual solution y if $y_U > 0$.



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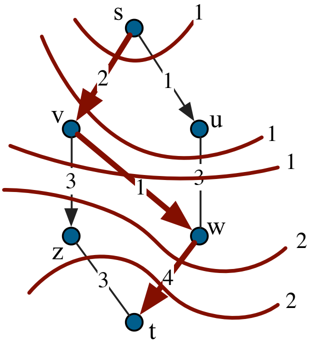
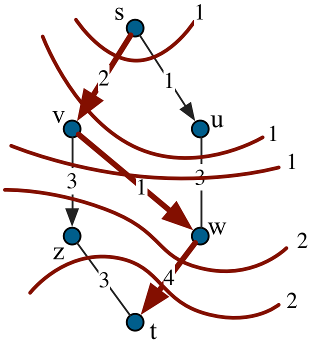
Example: $\delta(\{s, v, u\})$ is active, while $\delta(\{s, v\})$ is not!



Revisited: Shortest Path Optimality Conditions

Proposition

Let y be a feasible dual solution, and P and s, t-path. P is a shortest path if



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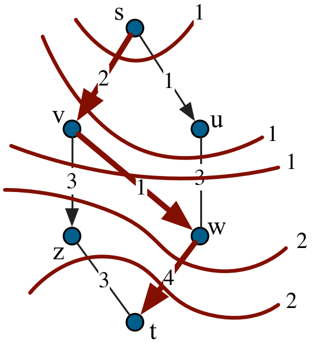

Revisited: Shortest Path Optimality Conditions

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Let y be a feasible dual solution, and P and s, t-path. P is a shortest path if

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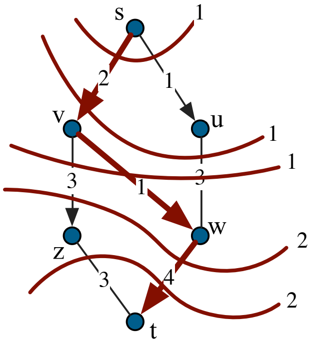

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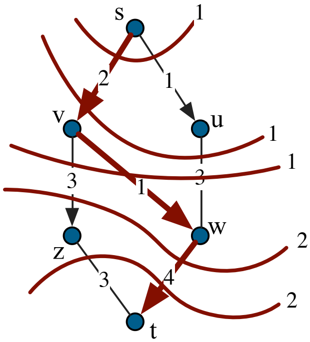

Note: Both conditions are satisfied in the example on the right.

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$$\sum_{e \in P} c_e = \sum_{e \in P} (\sum (y_U \,:\, e \in \delta(U))$$

because every edge on P is an equality edge by (i).

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because every edge on P is an equality edge by (i). The right-hand side equals

$$\sum (y_U \cdot |P \cap \delta(U)| \,:\, \delta(U))$$

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Note: Both conditions are satisfied in the example on the right!

Proof: Let's suppose that P and y satisfy (i) and (ii) of the proposition. Then,

$$\sum_{e \in P} c_e = \sum_{e \in P} (\sum (y_U \ : \ e \in \delta(U))$$

because every edge on P is an equality edge by (i). The right-hand side equals

$$\sum (y_U \cdot |P \cap \delta(U)| \ : \ \delta(U))$$

But, by (ii), $y_U > 0$ only if $|P \cap \delta(U)| = 1$. Hence:

$$\sum_{e \in P} c_e = \sum_U y_U$$

$\square$

Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

$y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$
while $t \notin U$ **do**
    Let ab be an edge in $\delta(U)$ of smallest slack for y where $a \in U, b \notin U$
    $y_U := \text{slack}_y(ab)$
    $U := U \cup \{b\}$
    change edge ab into an arc $\overrightarrow{ab}$
end while
return A directed st-path P.

Note: The algorithm terminates since one vertex is added to U in every step and V is finite.

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It suffices to show:

Proposition

The Shortest Path Algorithm maintains throughout its execution that:

(I1) y is a feasible dual,

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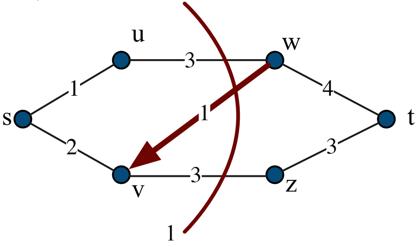

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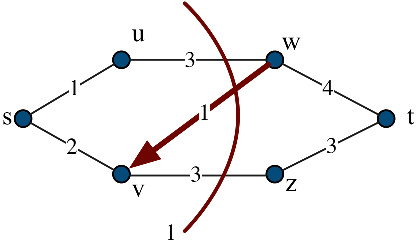

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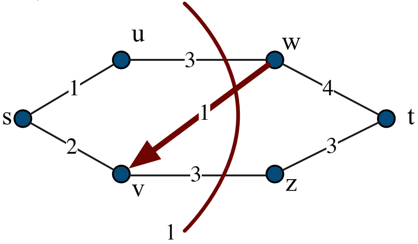

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(I4) for every $u \in U$ there is a directed s,u-path, and

(I5) arcs have both ends in U.

Correctness of the Shortest Path Algorithm

Suppose the invariants hold when the algorithm terminates. Then:

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The Shortest Path Algorithm maintains throughout its execution that:

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Correctness of the Shortest Path Algorithm

Suppose the invariants hold when the algorithm terminates. Then:

- $t \in U$ and (I4) implies that there is a directed s, t-path P,

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Correctness of the Shortest Path Algorithm

Suppose the invariants hold when the algorithm terminates. Then:

- $t \in U$ and (I4) implies that there is a directed s,t-path P,
- y is feasible by (I1), and

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The Shortest Path Algorithm maintains throughout its execution that:

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(I3) no active cut $\delta(U)$ has an entering arc: an arc wu with $w \notin U$, and $u \in U$,

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Correctness of the Shortest Path Algorithm

Suppose the invariants hold when the algorithm terminates. Then:

- $t \in U$ and (I4) implies that there is a directed s, t-path P,
- y is feasible by (I1), and
- arcs on P are equality arcs by (I2)

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(I5) arcs have both ends in U.

Correctness of the Shortest Path Algorithm

Suppose the invariants hold when the algorithm terminates. Then:

- $t \in U$ and (I4) implies that there is a directed s, t-path P,
- y is feasible by (I1), and
- arcs on P are equality arcs by (I2)

To show: $\delta(U)$ active $\longrightarrow$ P has exactly one edge in $\delta(U)$.

Proposition

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Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

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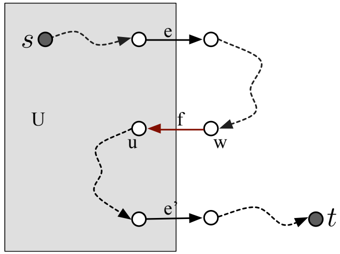
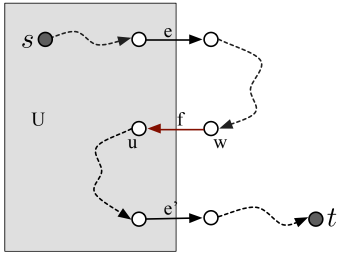


Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

Let e and e' be the first two edges on P that leave $\delta(U)$.

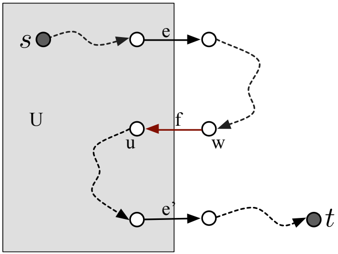


Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

Let e and e' be the first two edges on P that leave $\delta(U)$.

Then, there must also be an arc f on P that enters U,

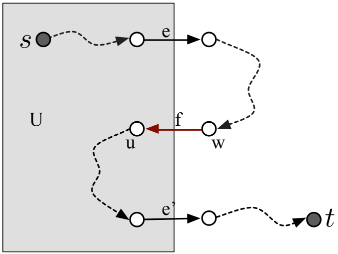


Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active and P has more than one edge in $\delta(U)$

Let e and e' be the first two edges on P that leave $\delta(U)$.

Then, there must also be an arc f on P that enters U, but this contradicts (I3)!



Proposition

The Shortest Path Algorithm maintains throughout its execution that:

(I3) no active cut $\delta(U)$ has an entering arc: an arc wu with $w \notin U$, and $u \in U$

Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

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Let's now prove the proposition!

Proposition

The Shortest Path Algorithm maintains throughout its execution that:

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Let's now prove the proposition!

Trivial: (I1) – (I5) hold after Step 1.

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Let's now prove the proposition!

Trivial: (I1) – (I5) hold after Step 1.
Suppose (I1) – (I5) hold before Step 3.

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Let's now prove the proposition!

Trivial: (I1) – (I5) hold after Step 1.
Suppose (I1) – (I5) hold before Step 3.
We will show that they also hold after Step 6.

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Shortest path dual:

$$\max \quad \sum(y_S \; : \; \delta(S) \; s,t\text{-cut})$$

$$\text{s.t.} \quad \sum(y_S \; : \; e \in \delta(S)) \leq c_e \qquad (e \in E)$$

$$y \geq \mathbb{0}$$

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$$y \geq \mathbb{0}$$

Note: In Step 3-6, only y_U for the current U changes.

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Shortest path dual:

$$\begin{aligned} \max \quad & \sum (y_S \ : \ \delta(S) \ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S \ : \ e \in \delta(S)) \leq c_e \\ & \qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

Note: In Step 3-6, only y_U for the current U changes.

y_U appears only on the left-hand sides of constraints for edges in $\delta(U)$.

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$$y \geq \mathbb{0}$$

Note: In Step 3-6, only y_U for the current U changes.

y_U appears only on the left-hand sides of constraints for edges in $\delta(U)$.

The smallest slack of any of these constraints is precisely the increase in y_U.

(I1) y is Dual Feasible

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Note: In Step 3-6, only y_U for the current U changes.

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The smallest slack of any of these constraints is precisely the increase in y_U.

$\longrightarrow$ y remains feasible!

(I1) y is Dual Feasible

Algorithm 3.2 Shortest path.

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Note: In Step 3-6, only y_U for the current U changes.

y_U appears only on the left-hand sides of constraints for edges in $\delta(U)$.

The smallest slack of any of these constraints is precisely the increase in y_U.

$\longrightarrow$ y remains feasible!

Also: The constraint of the newly created arc holds with equality after the increase

(I1) y is Dual Feasible

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6: change edge ab into an arc $\overrightarrow{ab}$
7: **end while**
8: **return** A directed st-path P.

Shortest path dual:

$$\max \quad \sum (y_S \ : \ \delta(S) \ s,t\text{-cut})$$

$$\text{s.t.} \quad \sum (y_S \ : \ e \in \delta(S)) \leq c_e \quad (e \in E)$$

$$y \geq \mathbb{0}$$

Note: In Step 3-6, only y_U for the current U changes.

y_U appears only on the left-hand sides of constraints for edges in $\delta(U)$.

The smallest slack of any of these constraints is precisely the increase in y_U.

$\longrightarrow$ y remains feasible!

Also: The constraint of the newly created arc holds with equality after the increase
$\longrightarrow$ (I2) continues to hold and constraints for arcs have slack 0.

Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

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y_W := 0 for all st-cuts δ(W). Set U := {s}
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    U := U ∪ {b}
    change edge ab into an arc →ab
end while
return A directed st-path P.
```

- the only new active cut created is $\delta(U)$

Proposition

The Shortest Path Algorithm maintains throughout its execution that:

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Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

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- the only new active cut created is $\delta(U)$
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The Shortest Path Algorithm maintains throughout its execution that:

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8: **return** A directed st-path P.

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$\longrightarrow$ (I3) holds after Step 6

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Correctness of the Shortest Path Algorithm

Note: Algorithms adds arc ab in current step, and (I4) implies that there is a directed s, a-path P.

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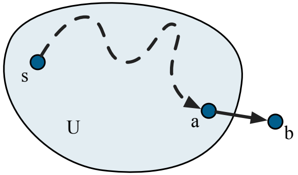

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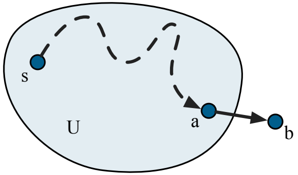

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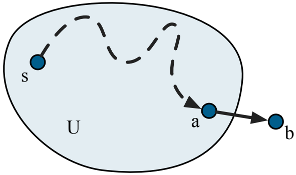

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$\longrightarrow$ since b is outside U, it cannot be on P, and thus, P together with ab is a directed s, b-path

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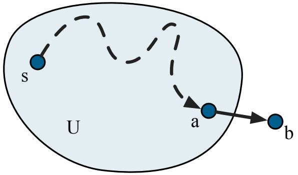

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$\longrightarrow$ (I4) holds at the end of loop

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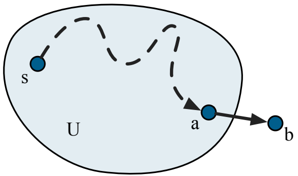

Finally, the only new arc added is ab. As b is added to U, (I5) continues to hold.

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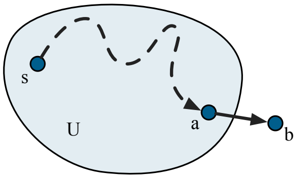

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We are now done!

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- We saw that the shortest path algorithm
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- Implicitly, we therefore showed that the shortest path LP always has an optimal integer solution!