Module 6: Nonlinear Programs (Convexity)

Definition

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This is a very general model, but NLPs can be very hard to solve!

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$$\boxed{\begin{array}{lrcl} \min & x_2 & & \\ \text{s.t.} & & & \\ & -x_1^2 - x_2 + 2 & \le & 0 \\ & x_2 - \frac{3}{2} & \le & 0 \\ & x_1 - \frac{3}{2} & \le & 0 \\ & -x_1 - 2 & \le & 0 \end{array}}$$

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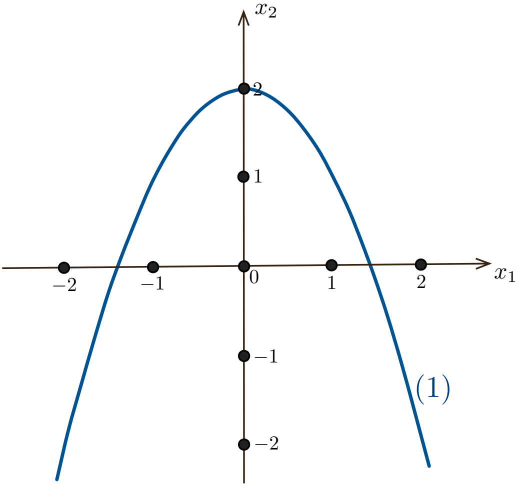

(1) $x_2 \geq 2 - x_1^2$.

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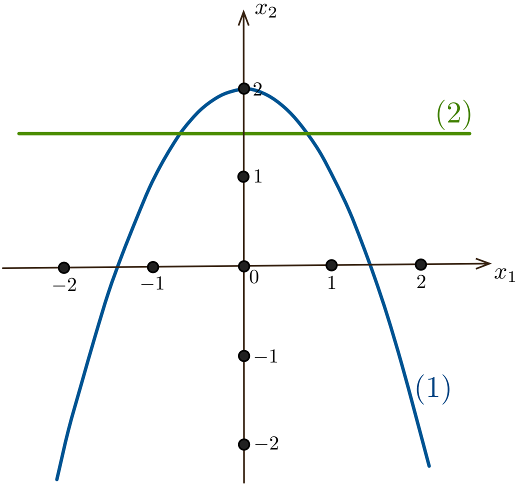

(1) $x_2 \geq 2 - x_1^2$.

(2) $x_2 \leq \frac{3}{2}$.

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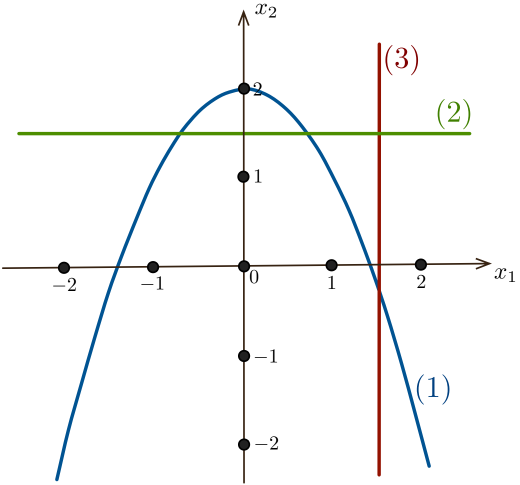

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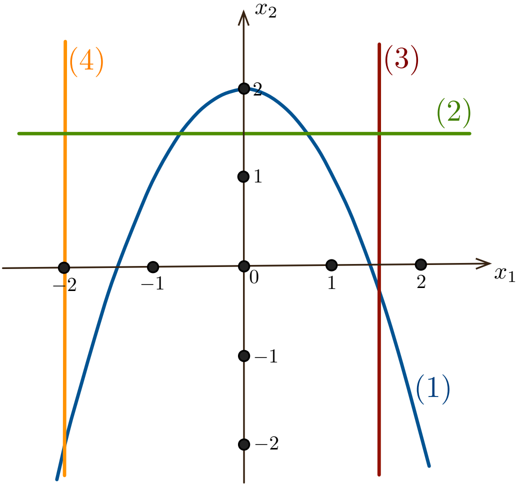

(1) $x_2 \geq 2 - x_1^2$.

(2) $x_2 \leq \frac{3}{2}$.

(3) $x_1 \leq \frac{3}{2}$.

(4) $x_1 \geq -2$.

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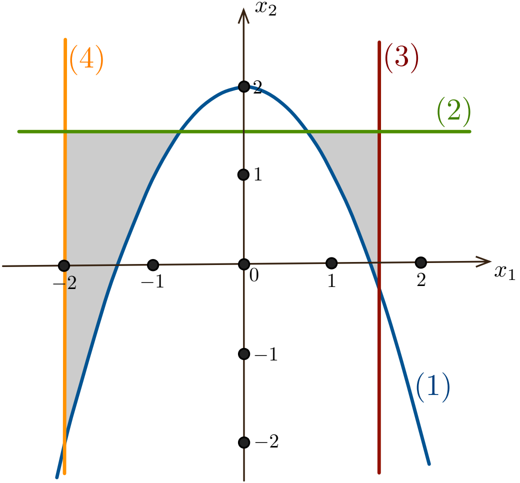

Feasible region

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We may assume $f(x)$ is a linear function, i.e., $f(x) = c^\top x$.

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We can rewrite (P) as

$$\boxed{\begin{array}{lll} \min & \lambda & \\ \text{s.t.} & & \\ & \lambda \geq f(x) & \\ & g_i(x) \leq 0 & \quad (i = 1, \ldots, k) \end{array}} \tag{Q}$$

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The optimal solution to (Q) will have $\lambda = f(x)$.

Nonlinear Programs Generalize Linear Programs

$$\max \quad x_1 + x_2$$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &= 4 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Nonlinear Programs Generalize Linear Programs

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\end{array}
$$

Nonlinear Programs can also generalize Integer Programs!

Nonlinear Programs Generalize Integer Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \in \{0,1\} \quad (j = 1, \ldots, n) \end{array}$$

$0,1$ IP

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Idea

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$x_j \in \{0,1\} \qquad \Longleftrightarrow \qquad x_j(1-x_j) = 0$

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Quadratic NLP

Remark

$0, 1$ IPs are hard to solve; thus, quadratic NLPs are also hard to solve.

Nonlinear Programs Generalize **Integer** Programs

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pure IP

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x_j integer $\iff$ $\sin(\pi x_j) = 0$.

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Remark

IPs are hard to solve; thus, NLPs are also hard to solve.

Question

What makes solving an NLP hard?

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$\boxed{\textsc{Meta strategy for solving an optimization problem}}$

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META STRATEGY FOR SOLVING AN OPTIMIZATION PROBLEM

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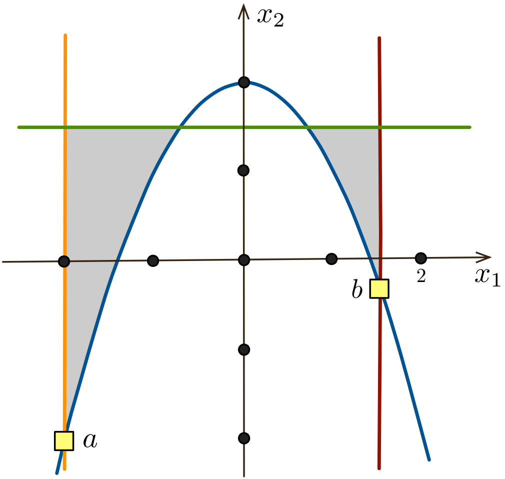

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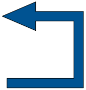
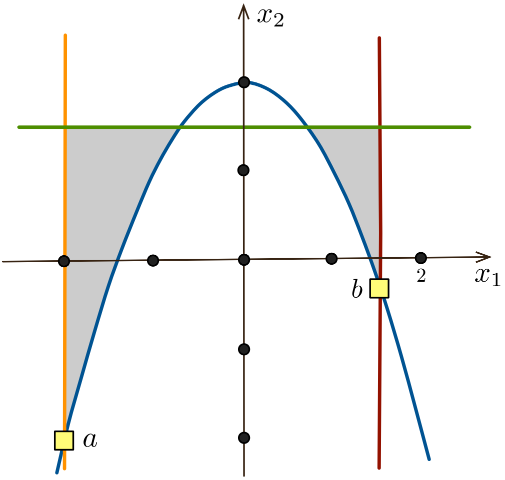


a is an optimal solution



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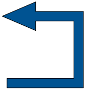

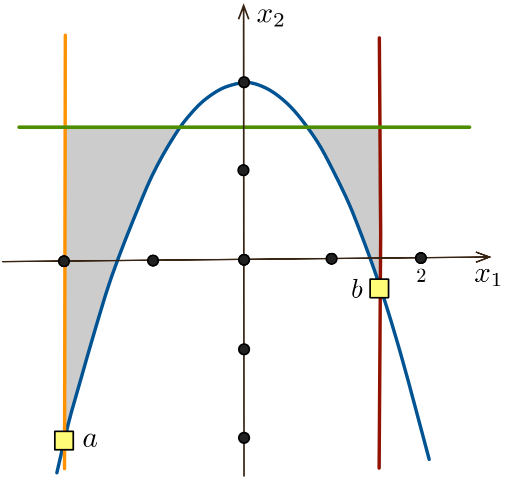

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no better solution around b

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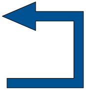

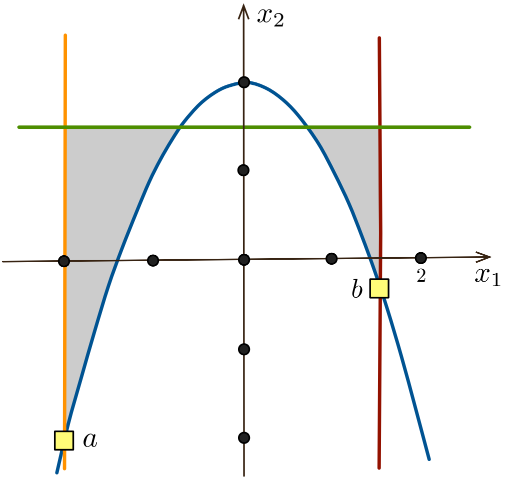

a is an optimal solution

no better solution around b

b is a local optimum

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$$\min\left\{f(x) : x \in S\right\}. \tag{P}$$

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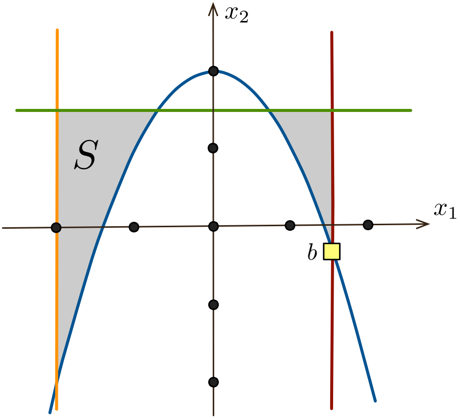

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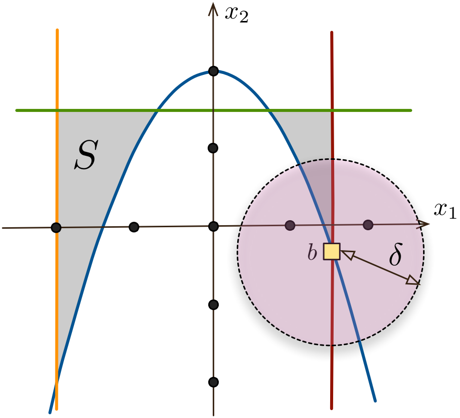

Convexity Helps

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Proposition

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Proposition

Consider

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If S is convex and x is a local optimum, then x is optimal.

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If S is convex and x is a local optimum, then x is optimal.

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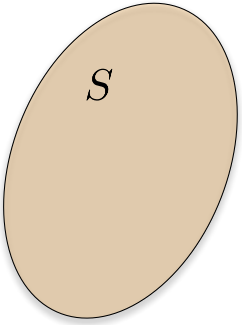

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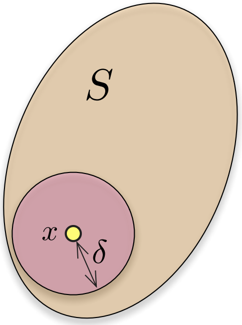

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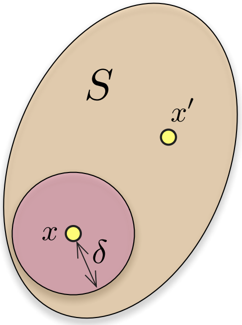

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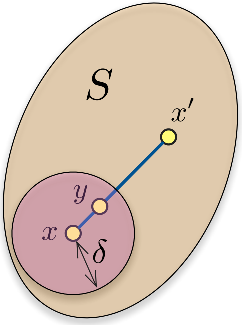

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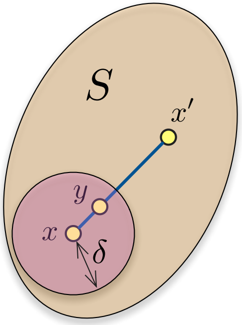

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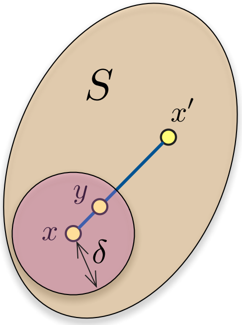

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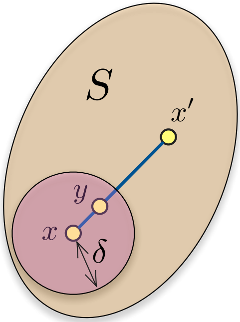

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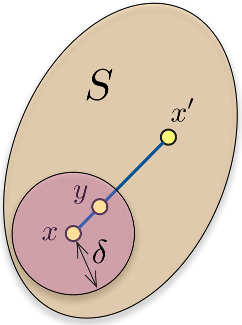

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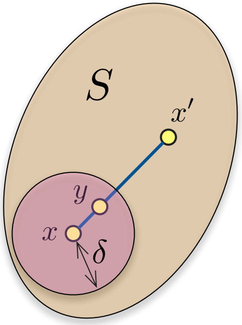

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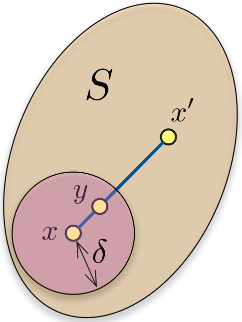

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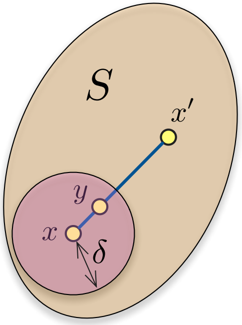

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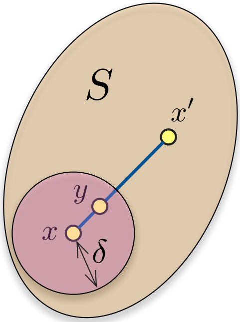

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- We will prove

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If $g_1, \ldots, g_k$ are all convex, then the feasible region of (P) is convex.

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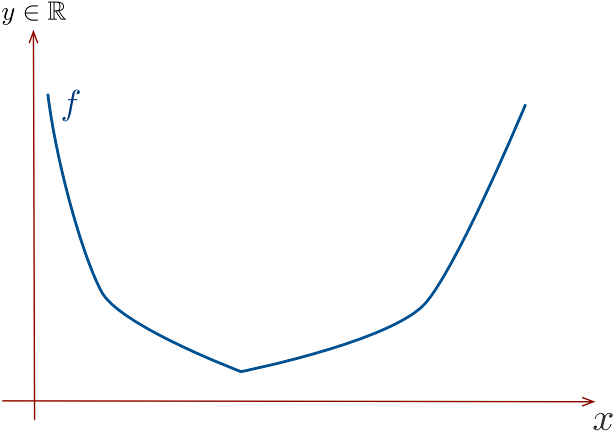

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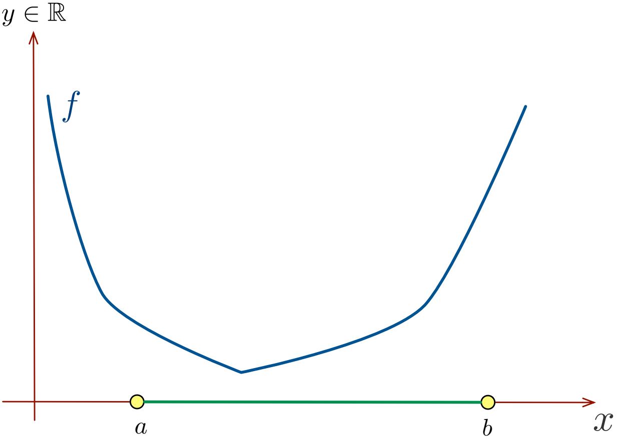

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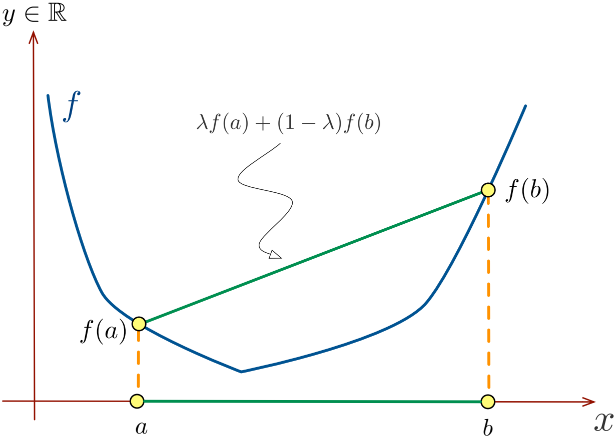

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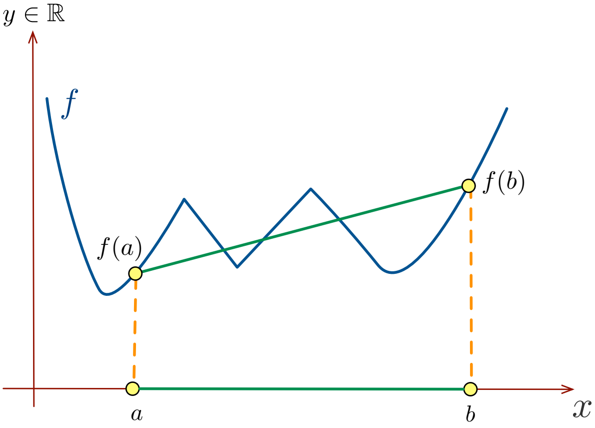

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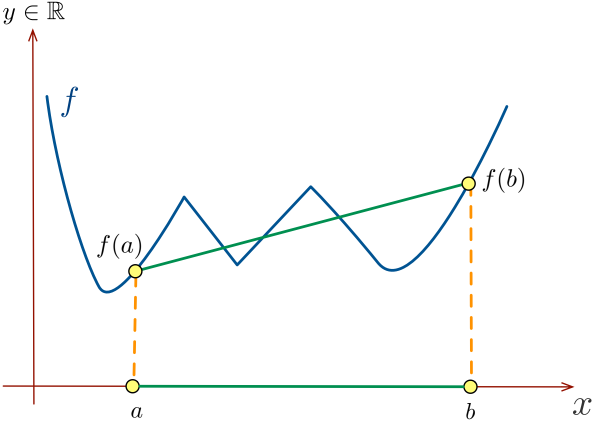

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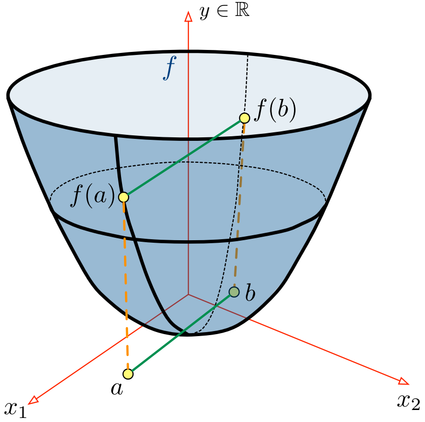

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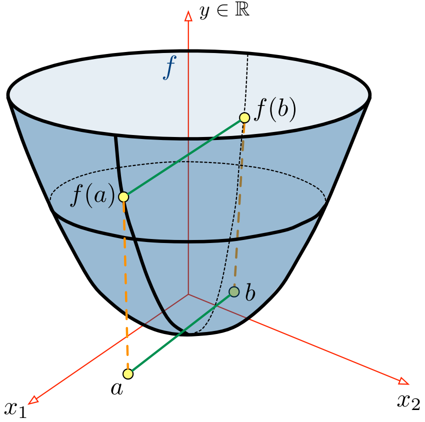

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which is the case as $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$.

Why Do We Care About Convex Functions?

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Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

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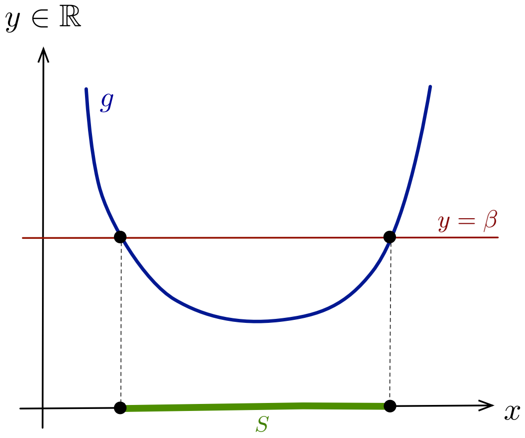

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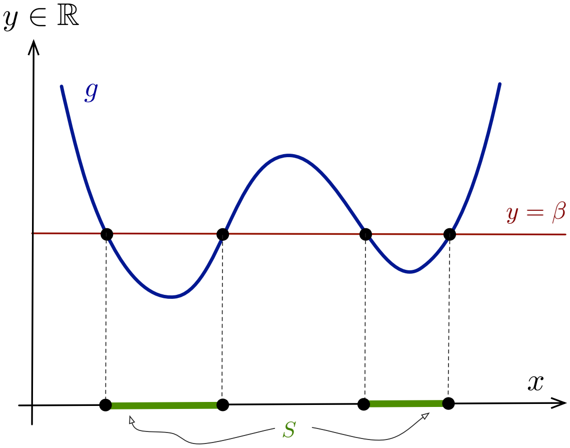

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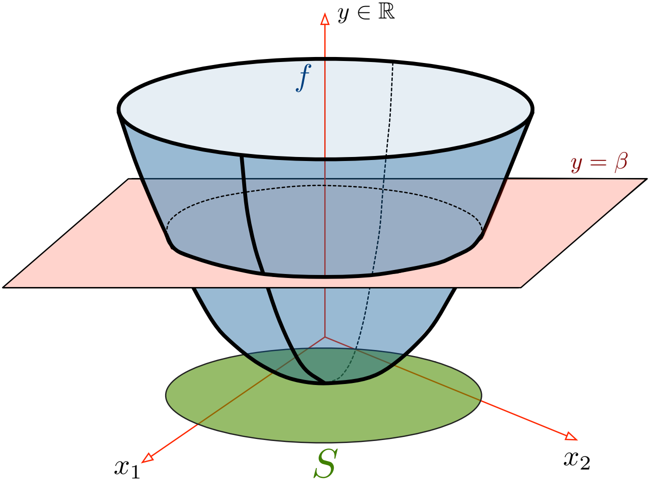

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Since the intersection of convex sets is convex, the result follows.

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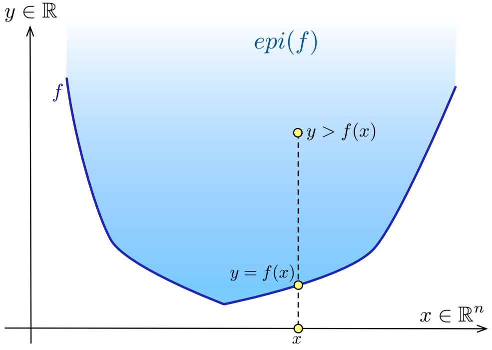

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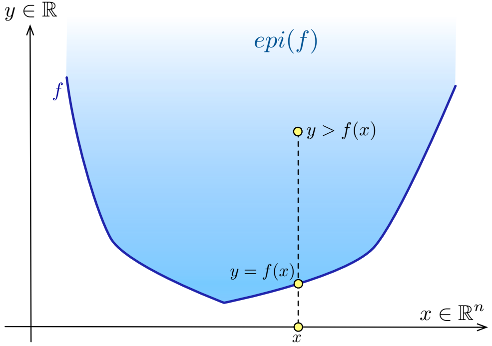

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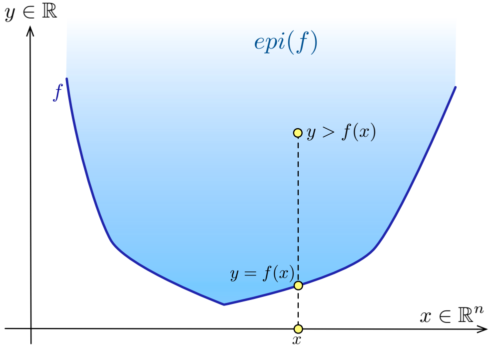

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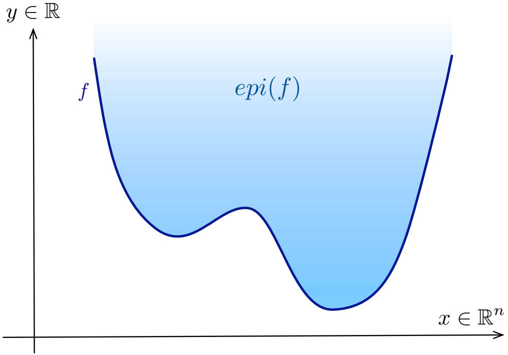

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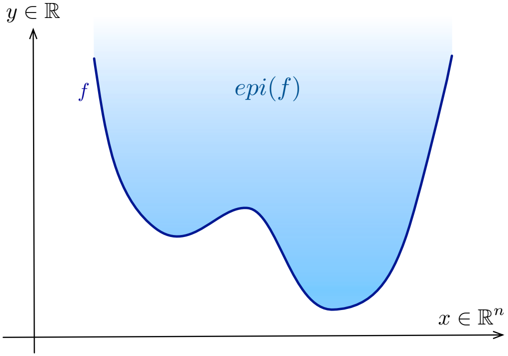

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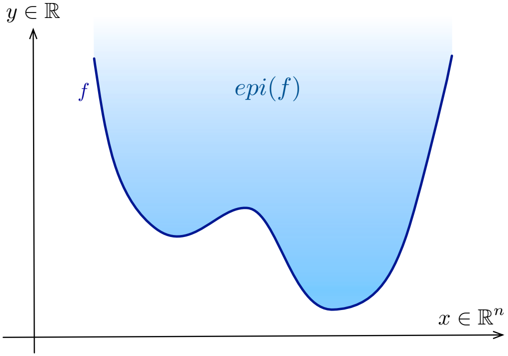

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Thus $(\star)$ is in $epi(f)$.

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