Assignment 4

Discussed during the tutorial on Thursday, November 17th, 2022

4.1 (10 points) Let G be a graph with distinct vertices $s, t \in V(G)$. An *even s-t-path* in G is a path in G with an even number of edges. Show that we can formulate the problem of finding an *even s-t-path* with as few edges as possible, as a minimum-cost perfect matching problem.
Hint: make a copy of the graph G with vertices s, t removed; call the resulting graph G'. Construct a new graph H starting with the union of graph G and G' an by joining every vertex $v \in V(H) \setminus \{s, t\}$ in G with its copy v' in G'.

4.2 (10 points) Let A be an $m \times n$ matrix and let b be a vector with m entries. Prove or disprove each of the following statements (in both cases, y is a vector with m entries):

(i) If there exists a y such that $y^T A \geq 0^T$ and $b^T y < 0$, then $Ax \leq b, x \geq 0$ has no solution.

(ii) If there exists some $y \geq 0$ such that $y^T A \geq 0^T$ and $b^T y < 0$, then $Ax \leq b, x \geq 0$ has no solution.

4.3 (10 points) Consider the following two LPs. In each one of them, starting from a feasible solution $\bar{x}$, construct a feasible solution x' with value larger than that of $\bar{x}$ by increasing as much as possible the value of exactly one of $\bar{x}_j$ for given j, keeping the other variable unchanged.

(a)

$$\begin{aligned} \max \ & (-1, 0, 0, 2)x \\ \text{subject to } & \begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x \geq 0 \ . \end{aligned}$$

Let $\bar{x} = (0, 2, 3, 0)^T$; change exactly one of $\bar{x}_1$ and $\bar{x}_4$.

(b)

$$\begin{aligned} \max \ & (0, 0, 4, -6)x \\ \text{subject to } & \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x \geq 0 \ . \end{aligned}$$

Let $\bar{x} = (2, 1, 0, 0)^T$; change one of $\bar{x}_3$ and $\bar{x}_4$.

4.4 (10 points) An important application of nonlinear optimization is portfolio optimization. In a fundamental version of such a problem, we usually have a fixed amount of capital that we wish to invest into several investment options. The usual goal is to maximize the return from this investment while ensuring that the risk of the constructed portfolio is small. Usually, stock that have the highest potential returns are the most volatile, and hence the most risky stocks to invest at the same time. The risk-averse investor will therefore have to strike a balance between return and risk. One way to achieve this balance is to attempt to minimize the overall volatility while guaranteeing a minimum expected return.

More concretely, imagine you had €500 available for investment into three different stocks 1, 2, and 3. In the following, we let S_i be the random variable for the annual return on €1 invested into stock i.

Assume that we know the expected annual return and its variance for each of the given stocks. The following table shows expected values and variances for the given variables:

i	$\mathbb{E}[S_i]$	$\mathsf{var}[S_i]$
1	0.1	0.2
2	0.13	0.1
3	0.08	0.15

In addition to the information above, we are also given the following covariances:

$$\mathsf{cov}(S_1, S_2) = 0.03, \mathsf{cov}(S_1, S_3) = 0.04, \mathsf{cov}(S_2, S_3) = 0.01 \ .$$

Recall that, for a random variable ψ,

$$\mathsf{var}(\psi) = \mathbb{E}[(\psi - \mathbb{E}[\psi])^2],$$

and for a pair of random variables ψ_1, ψ_2,

$$\mathsf{cov}(\psi_1, \psi_2) = \mathbb{E}[(\psi_1 - \mathbb{E}[\psi_1])(\psi_1 - \mathbb{E}[\psi_2])] \ .$$

The goal is now to invest the given €500 into the three stocks such that the expected return is at least €50, and such that the total variance of our investment is minimized.

Introduce a variable x_i denoting the amount of money invested into stock i for all $i \in \{1, 2, 3\}$. Using these variables, formulate the problem as a quadratic optimization problem such that the objective function is a quadratic function of x_i, and all the constraints are linear.

Upload your solutions as a `.pdf`-file to the course page on the TUHH e-learning portal until Tuesday, 8am on 15th November, 2022.