Introduction to Optimization

Part 1: Formulations (Overview)

Outline

Introducing Optimization

Three Case Studies

A Modeling Example

Optimization - Abstract Perspective

- Abstractly, we will focus on problems of the following form:
  - Given: set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$
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- Very general problem that is enormously useful in virtually very branch of industry.
- Bad news: the above problem is notoriously hard to solve (and may not even be well-defined).

Optimization - Important Special Cases

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  (C) Nonlinear Programming. A is given by *non-linear* constraints, and f is a *non-linear* function.

Optimization - Typical Workflow

Typical development process has three stages.

- Starting point is English language description of practical problem

Practical Problem:
- Description in plain language
- Supporting data

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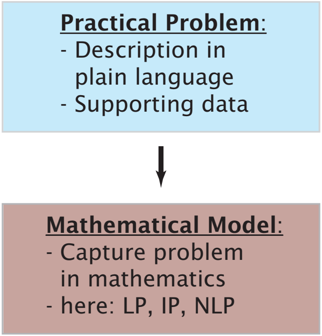

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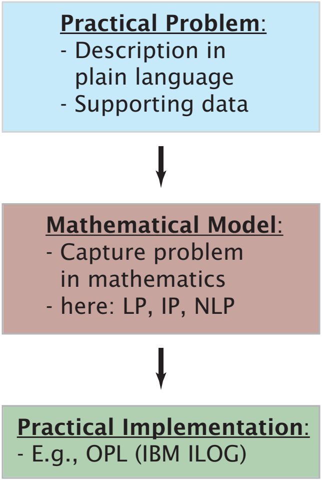

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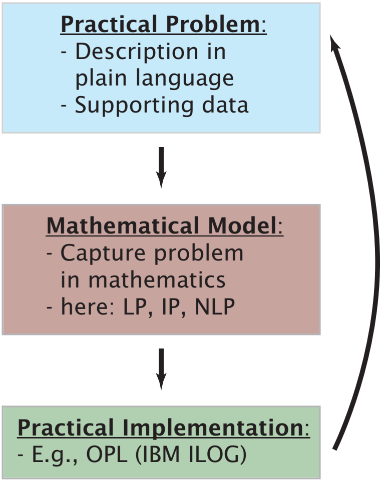
- Iterate!



Optimization in Practice

Optimization is everywhere! Some examples:

- Booking hotel rooms or airline tickets,
- Setting the market price of a kwh of electricity,
- Determining an “optimal” portfolio of stocks,
- Computing energy efficient circuits in chip design,
- and many more!

CSX Rail

- One of the largest transport suppliers in the United States.
- CSX operates 21000 miles of rail network
- 11 Billion in annual revenue
- Serves 23 states, Ontario and Quebec
- Operates 1200 trains per day

- Has a fleet of 3800 locomotives, and more than 100000 freight cars
- Transports 7.4 million car loads per year

Optimization @ CSX Rail

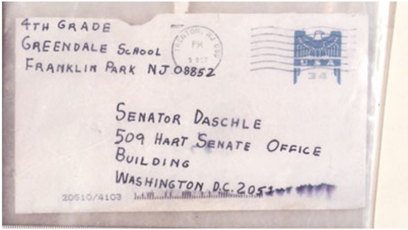
- [Acharya, Sellers, Gorman '10] use mathematical programming to optimally allocate and reposition empty railcars dynamically

- Implementing system yields the following estimated benefits for CSX:

  Annual savings: $51 million
  Avoided rail car capital investment: $1.4 billion

Optimization in Disease Control

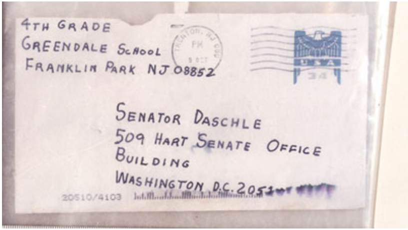
- [Lee et al. '13] Use mathematical programming to prepare for disease outbreak and medical catastrophes.
- Where should we place medical dispensing facilities, and how should we staff these in order to disseminate medication as quickly as possible to population?
- How should dispensing be scheduled?



2001 Anthrax letter sent to Senator T. Daschle

Optimization in Disease Control

- In collaboration with the Center for Disease Control, [Lee et al. '13] develop decision support suite RealOpt
- Suite is being used by $\approx$ 6500 public health and emergency directors in the USA to design, place and staff medical dispensing centers
- In tests, throughput in medical dispensing centers increases by several orders of magnitude.



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E.g.: producing a unit of product 3 requires 6h on machine 1, 5h on machine 2, 5h of skilled, and 7h of unskilled labour. It can be sold at $220 per unit.

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Restrictions:

- WaterTech has available 700h of machine 1, and 500h of machine 2 time
- Can purchase 600h of skilled labour at $8 per hour, and at most 650h of unskilled labour at $6 per hour

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Formulate this as a mathematical program!

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- Similarly, we may not use more than 500h of machine 2 time:

$$\implies \quad 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500$$

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- Producing x_i units of product $i \in \mathcal{P}$ requires

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  units of skilled labour, and this must not exceed y_s.

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- …and $y_s \leq 600$ as well as $y_u \leq 650$ as only limited amounts of labour can be purchased.

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- Revenue from sales:

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$$\begin{aligned} \text{maximize} \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 \\ & - 8y_s - 6y_u \end{aligned}$$

WaterTech – Entire Model

$$
\begin{array}{rl}
\max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t.} & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
& 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\
& 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\
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\end{aligned}
$$

Solution (via CPLEX): $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$, $y_u = 650$ of profit \$15433 + $\frac{1}{3}$.

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To clarify these ideas let us consider a simple example. Suppose WaterTech manufactures four products, requiring time on two machines and two types (skilled and unskilled) of labour. The amount of machine time and labor (in hours) needed to produce a unit of each product and the sales prices in dollars per unit of each product are given in the following table:

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- Similar: solution to word description is an assignment to the unknowns

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- It is easily checked that

$$x_1 = 10, x_2 = 50, x_3 = 0, x_4 = 20, y_s = y_u = 600$$

is feasible for the mathematical program we wrote.

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- In the example, the map was simply the identity. This need not necessarily be the case in general!