

Module 1: Formulations (IP Models)

Recap: WaterTech

$$\begin{aligned} \max \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\ \text{s.t.} \quad & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\ & 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\ & 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\ & 7x_1 + 8x_2 + 7x_3 + 4x_4 \leq y_u \\ & y_s \leq 600 \\ & y_u \leq 650 \\ & x_1, x_2, x_3, x_4, y_u, y_s \geq 0. \end{aligned}$$

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Optimal solution: $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$,
 $y_u = 650$

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Optimal solution: $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$,
 $y_u = 650$

Fractional solutions are often not desirable! **Can we force solution to take on only integer values?**

- Yes!

An **integer program** is a linear program with **added integrality constraints** for some/all variables.

$$\begin{array}{ll}\max & x_1 + x_2 + 2x_4 \\ \text{s.t.} & x_1 + x_2 \leq 1 \\ & -x_2 - x_3 \geq -1 \\ & x_1 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

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- We call an IP **mixed** if there are **integer and fractional** variables, and **pure** otherwise.

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- We call an IP **mixed** if there are **integer and fractional** variables, and **pure** otherwise.
- Difference between **LPs** and **IPs** is **subtle**. Yet: LPs are **easy to solve**, IPs are **not!**

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 \text{s.t.} & x_1 + x_2 \leq 1 \\
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- The **running time** of an algorithm is then the number of **steps** that an algorithm takes.

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- Every problem instance has a **size** which we normally denote by n .
Think: $n \sim$ number of variables/constraints of IP.
- The **running time** of an algorithm is then the number of **steps** that an algorithm takes.
- It is stated as **a function of n** : $f(n)$ measures the **largest** number of steps an algorithm takes on an instance **of size n** .

Can we solve IPs?

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- LPs can be solved efficiently.

The New York Times U.S.

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BREAKTHROUGH IN PROBLEM SOLVING

By JAMES GLEICK
Published: November 19, 1984

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

These problems are fiendishly complicated systems, often with

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- IPs are very unlikely to admit efficient algorithms!

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- LPs can be solved efficiently.
- IPs are very unlikely to admit efficient algorithms!
- It is very important to look for an efficient algorithm for a problem. The table states actual running times of a computer that can **execute 1 million** operations per second on an **instance of size $n = 100$** :

$f(n)$	n	$n \log_2(n)$	n^3	1.5^n	2^n
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IP Models: Knapsack

KitchTech Shipping

- Company wishes to **ship crates** from Toronto to Kitchener.

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- Each crate type has **weight** and **value**:

Type	1	2	3	4	5	6
weight (lbs)	30	20	30	90	30	70
value (\$)	60	70	40	70	20	90

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- **Total weight** of crates shipped must not exceed 10,000 lbs.

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weight (lbs)	30	20	30	90	30	70
value (\$)	60	70	40	70	20	90

- **Total weight** of crates shipped must not exceed 10,000 lbs.
- **Goal:** Maximize total value of shipped goods.

IP Model

- Variables.

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$$\text{s.t.} \quad 30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000$$

$$x_i \geq 0 \quad (i \in [6])$$

$$x_i \text{ integer} \quad (i \in [6])$$

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Let's make this model a bit more interesting...

KitchTech: Added Conditions

Suppose that ...

- We must not send more than 10 crates of the same type.

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- We must not send more than 10 crates of the same type.
- Can only send crates of type 3, if we send at least 1 crate of type 4.

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- $x_4 \geq 1 \rightarrow$ new constraint is redundant!

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- $x_4 = 0 \rightarrow$ new constraint becomes

$$x_3 \leq 0.$$

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KitchTech: One More Tricky Case

Suppose that we must

- ① take a total of at least 4 crates of type 1 or 2, or
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Ideas?

Create a new variable y

s.t.

- ① $y = 1 \longrightarrow$
 $x_1 + x_2 \geq 4,$
- ② $y = 0 \longrightarrow$

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$$\textcircled{1} \quad y = 1 \longrightarrow \\ x_1 + x_2 \geq 4,$$

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Force y to take on values
0 or 1.

Add constraints:

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$$\textcircled{1} \quad x_1 + x_2 \geq 4y$$

$$\textcircled{2} \quad x_5 + x_6 \geq 4(1 - y)$$

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$$x_3 \leq 10x_4$$

$$x_1 + x_2 \geq 4y$$

$$x_5 + x_6 \geq 4(1 - y)$$

$$0 \leq y \leq 1$$

$$0 \leq x_i \leq 10 \quad (i \in [6])$$

y integer

x_i integer $(i \in [6])$

Binary Variables

Variable y is called a **binary variable**.

These are **very useful** for modeling **logical** constraints of the form:

[**Condition (A or B) and C**] \rightarrow D

Will see more examples ...

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 & x_5 + x_6 \geq 4(1 - y) \\
 & 0 \leq y \leq 1 \\
 & 0 \leq x_i \leq 10 \quad (i \in [6]) \\
 & y \text{ integer} \\
 & x_i \text{ integer} \quad (i \in [6])
 \end{aligned}$$

IP Models: Scheduling

- The neighbourhood coffee shop is **open on workdays**.



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- Daily demand for workers:

Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7



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Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7

- Each worker **works for 4 consecutive days** and has one day off.



- The neighbourhood coffee shop is **open on workdays**.
- Daily demand for workers:

Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7

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Can we solve this using IP?

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All **but those that start on Tuesday**; i.e.,

$$x_M + x_W + x_{Th} + x_F.$$

Constraints

[Daily Demand]

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Monday:

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$$x_M + x_T + x_{Th} + x_F \geq 5$$

Constraints

[Daily Demand]

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Monday:

$$x_M + x_W + x_{Th} + x_F \geq 3$$

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$$x_M + x_T + x_{Th} + x_F \geq 5$$

Wednesday:

$$x_M + x_T + x_W + x_F \geq 9$$

Constraints

[Daily Demand]

Mon	Tues	Wed	Thurs	Fri
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Tuesday:

$$x_M + x_T + x_{Th} + x_F \geq 5$$

Wednesday:

$$x_M + x_T + x_W + x_F \geq 9$$

Thursday:

$$x_M + x_T + x_W + x_T \geq 2$$

Constraints

[Daily Demand]

Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7

Monday: $x_M + x_W + x_{Th} + x_F \geq 3$

Tuesday: $x_M + x_T + x_{Th} + x_F \geq 5$

Wednesday: $x_M + x_T + x_W + x_F \geq 9$

Thursday: $x_M + x_T + x_W + x_T \geq 2$

Friday: $x_T + x_W + x_{Th} + x_F \geq 7$

Scheduling LP

$$\begin{array}{ll} \min & x_M + x_T + x_W + x_{Th} + x_F \\ \text{s.t.} & x_M + x_W + x_{Th} + x_F \geq 3 \\ & x_M + x_T + x_{Th} + x_F \geq 5 \\ & x_M + x_T + x_W + x_F \geq 9 \\ & x_M + x_T + x_W + x_T \geq 2 \\ & x_T + x_W + x_{Th} + x_F \geq 7 \\ & x \geq 0, x \text{ integer} \end{array}$$

Quiz

Given an integer program with integer variables x_1, \dots, x_6 . Let

$$\mathcal{S} := \{127, 289, 1310, 2754\}.$$

We want to add constraints and/or variables to the IP that enforce that the $x_1 + \dots + x_6$ is in \mathcal{S} . How?

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- **Want:** Exactly one of these variables is 1 in a feasible solution.
- If $y_n = 1$ for $n \in \mathcal{S}$ then $\sum_{i=1}^6 x_i = n$

Quiz

Add the following constraints:

$$y_{127} + y_{289} + y_{1310} + y_{2754} = 1$$

$$\sum_{i=1}^6 x_i = \sum_{i \in \mathcal{S}} iy_i$$

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Why is the resulting IP correct?

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- Binary variables are useful for expressing **logical conditions**.