Module 2: Linear Programs (Finding a Feasible Solution)

The Problem

Consider

$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}.$$

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An easier question,

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How do we find a feasible solution?

These two questions are equivalent.

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$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}.$$

To run Simplex, we need a feasible basis.

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How do we find a feasible basis?

An easier question,

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How do we find a feasible solution?

These two questions are equivalent.

Exercise

There is an algorithm that, given a feasible solution, finds a feasible basis.

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$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}.$$

To run Simplex, we need a feasible basis.

Question

How do we find a feasible basis?

An easier question,

Question

How do we find a feasible solution?

These two questions are equivalent.

Exercise

There is an algorithm that, given a feasible solution, finds a feasible basis.

⇒ We will focus on the second question.

The Key Idea

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$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}$$

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Algorithm 1

INPUT: A, b, c, and a feasible solution

OUTPUT: Optimal solution/detect LP unbounded.

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Simplex +
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INPUT: A, b, c, and a feasible solution

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Simplex +
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Algorithm 2

INPUT: A, b, c.

OUTPUT: Feasible solution/detect there is none.

The Key Idea

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Algorithm 1

<u>INPUT:</u> A, b, c, and a feasible solution

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<u>INPUT:</u> A, b, c.

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HOW?

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INPUT: A, b, c.

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HOW?

We will show that...

The Key Idea

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INPUT: A, b, c, and a feasible solution

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OK
Simplex +
exercise

Algorithm 2

INPUT: A, b, c.

OUTPUT: Feasible solution/detect there is none.

HOW?

We will show that...

Proposition

We can use Algorithm 1 to get Algorithm 2.

A First Example

A First Example

Problem: Find a feasible solution/detect none exist for

$$\max \quad (1, 2, -1, 3)x$$

s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

A First Example

Problem: Find a feasible solution/detect none exist for

$$\begin{array}{ll} \max & (1,2,-1,3)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Remark

It does not depend on the objective function.

<u>Problem:</u> Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

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Step 1. Multiply the equations such that the RHS is non-negative.

Problem: Find a feasible solution/detect none exist for

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Step 1. Multiply the equations such that the RHS is non-negative. OK

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\min \quad x_5 + x_6$$

s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

min $x_5 + x_6$

s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$x \geq \mathbf{0}$

x_5, x_6 are the auxiliary variables

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Step 1. Multiply the equations such that the RHS is non-negative. OK

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The auxiliary problem is

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Step 1. Multiply the equations such that the RHS is non-negative. OK

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s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

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Remark

The auxiliary problem is

- feasible, since $(0,0,0,0,7,13)^\top$ is a solution, and

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Step 1. Multiply the equations such that the RHS is non-negative. OK

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$$\begin{aligned} &\min \quad x_5 + x_6 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix}1 & 5 & 2 & 1 & 1 & 0\\ 2 & 9 & 0 & -3 & 0 & 1\end{pmatrix} x = \begin{pmatrix}7\\13\end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$

x_5, x_6 are the auxiliary variables

Remark

The auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and
- bounded, as 0 is the lower bound.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{aligned} \min \quad & x_5 + x_6 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

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Remark

The auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and
- bounded, as 0 is the lower bound.

⟹ Therefore, the auxiliary problem has an optimal solution.

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Step 3. Solve the auxiliary problem using Algorithm 1.

$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

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$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

since $x_5 = x_6 = 0$.

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$$\begin{aligned} \min \quad & x_5 + x_6 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

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$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

since $x_5 = x_6 = 0$.

➡ Therefore, $(2, 1, 0, 0)^\top$ is a feasible solution to $(\star)$.

A Second Example

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Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\min \quad z = x_4 + x_5$$

s.t.

$$\begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

A Second Example

Problem: Find a feasible solution/detect none exist for

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Step 3. Solve the auxiliary problem using Algorithm 1.

A Second Example

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$$\begin{aligned} \min \quad & z = x_4 + x_5 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

x_4, x_5 are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(0, 0, 1, 0, 3)^\top$ is an optimal solution to the auxiliary problem.

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\min \quad z = x_4 + x_5$$

s.t.

$$\begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

x_4, x_5 are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(0, 0, 1, 0, 3)^\top$ is an optimal solution to the auxiliary problem.

However, $(0, 0, 1)^\top$ is NOT a solution to $(\star)$.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{array}{ll} \min & z = x_4 + x_5 \\ \text{s.t.} & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\min \quad z = x_4 + x_5$$

s.t.

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$$x \geq \mathbf{0}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\min \quad z = x_4 + x_5$$

s.t.

$$\begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

min $\quad z = x_4 + x_5$

s.t.

$$\begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$x \geq \mathbf{0}$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

$$\min \quad z = x_4 + x_5$$

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optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x'_1, x'_2, x'_3.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

min $\quad z = x_4 + x_5$

s.t.

$$\begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x'_1, x'_2, x'_3.

Then, $(x'_1, x'_2, x'_3, 0, 0)^\top$ is a feasible solution to the auxiliary problem,

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

min $\quad z = x_4 + x_5$

s.t.

$$\begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$x \geq \mathbf{0}$

the auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x'_1, x'_2, x'_3.

Then, $(x'_1, x'_2, x'_3, 0, 0)^\top$ is a feasible solution to the auxiliary problem,

but that solution has of value 0. This is a contradiction.

Formalize

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \tag{$\star$}$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \begin{pmatrix} A & \mid & I \end{pmatrix} x = b \\ & x \geq \mathbf{0} \end{array}$$

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

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$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \begin{pmatrix} A & \vert & I \end{pmatrix} x = b \\ & x \geq \mathbf{0} \end{array}$$

$x_{n+1}, \ldots, x_{n+m}$ are the auxiliary variables

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \left(\; A \; \middle| \; I \; \right) x = b \\ & x \geq \mathbf{0} \end{array}$$

$x_{n+1}, \ldots, x_{n+m}$ are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

Formalize

Problem: Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct the auxiliary problem (A $m \times n$ matrix).

$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \left(\begin{array}{c|c} A & I \end{array} \right) x = b \\ & x \geq \mathbf{0} \end{array}$$

$x_{n+1}, \ldots, x_{n+m}$ are the auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$ is an optimal solution to the auxiliary problem.

Formalize

<u>Problem:</u> Find a feasible solution/detect none exist for

$$Ax = b \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

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If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to $(\star)$.

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Proposition

If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to $(\star)$.

Proof

When $z = 0$, we must have $x_{n+1} = \ldots = x_{n+m} = 0$.

Problem: Find a feasible solution/detect none exist for

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Proposition

When $z > 0$, then $(\star)$ has no solution.

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \left(\quad A \quad \middle| \quad I \quad \right) x = b \\ & x \geq \mathbf{0} \end{array}$$

the auxiliary problem

optimal solution
$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$

Proposition

When $z > 0$, then $(\star)$ has no solution.

$$Ax = b \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

min	$z = x_{n+1} + \ldots + x_{n+m}$
s.t.	
	$\begin{pmatrix} A & \vert & I \end{pmatrix} x = b$
	$x \geq \mathbf{0}$

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Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

$$Ax = b \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \left(\quad A \quad \middle| \quad I \quad \right) x = b \\ & x \geq \mathbf{0} \end{array}$$

the auxiliary problem

optimal solution
$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$

Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

$$Ax = b \quad \text{and} \quad x \geq \mathbf{0} \tag{$\star$}$$

min	$z = x_{n+1} + \ldots + x_{n+m}$
s.t.	
	$\begin{pmatrix} A & \vert & I \end{pmatrix} x = b$
	$x \geq \mathbf{0}$

the auxiliary problem

optimal solution
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When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

but that solution has of value 0.

$$Ax = b \quad \text{and} \quad x \geq \mathbf{0} \tag{$\star$}$$

$$\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \begin{pmatrix} A & \mid & I \end{pmatrix} x = b \\ & x \geq \mathbf{0} \end{array}$$

the auxiliary problem

optimal solution

$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$

Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is a feasible solution to the auxiliary problem,

but that solution has of value 0. This is a contradiction.

The 2-Phase Method

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To solve

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\},$$

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we proceed in two phases:

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Phase 1. Find a feasible solution/detect none exist.

Phase 2. Given that feasible solution:
find an optimal solution/detect LP unbounded.

The 2-Phase Method

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we proceed in two phases:

Phase 1. Find a feasible solution/detect none exist.

Phase 2. Given that feasible solution:
find an optimal solution/detect LP unbounded.

Example

Solve the following LP,

$$\begin{array}{ll}\max & (1,1,1)x \\ \text{s.t.} & \\ & \begin{pmatrix}1 & 2 & -1 \\ 1 & -1 & 1\end{pmatrix} x = \begin{pmatrix}4 \\ 4\end{pmatrix} \\ & x \geq \mathbf{0}\end{array}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\min \quad z = x_4 + x_5$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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NOT in SEF

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

max $\quad z = -x_4 - x_5$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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In SEF

Phase 1. Find a feasible solution/detect none exist for

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In SEF

feasible basis $B = \{4, 5\}$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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$x \geq \mathbf{0}$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

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$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

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In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

To rewrite $B = \{4, 5\}$ in canonical form, you can

Phase 1. Find a feasible solution/detect none exist for

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = -x_4 - x_5$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

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In SEF

feasible basis $B = \{4, 5\}$

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To rewrite $B = \{4, 5\}$ in canonical form, you can

- use the formulae, OR

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Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = -x_4 - x_5$$

s.t.

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In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

To rewrite $B = \{4, 5\}$ in canonical form, you can

- use the formulae, OR
- notice $A_B = I$ and rewrite the objective function as follows...

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = -x_4 - x_5$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$z = (0 \quad 0 \quad 0 \quad -1 \quad -1)x$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

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$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{aligned} z &= (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\ 0 &= (1 \quad 2 \quad -1 \quad 1 \quad 0)x - 4 \end{aligned}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = -x_4 - x_5$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{aligned} z &= (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\ 0 &= (1 \quad 2 \quad -1 \quad 1 \quad 0)x - 4 \\ 0 &= (1 \quad -1 \quad 1 \quad 0 \quad 1)x - 4 \end{aligned}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\begin{array}{ll} \max & z = -x_4 - x_5 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{array}{rcl} z &=& (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\ 0 &=& (1 \quad 2 \quad -1 \quad 1 \quad 0)x - 4 \\ 0 &=& (1 \quad -1 \quad 1 \quad 0 \quad 1)x - 4 \\ \hline z &=& (2 \quad 1 \quad 0 \quad 0 \quad 0)x - 8 \end{array}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = (2 \quad 1 \quad 0 \quad 0 \quad 0) - 8$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = (2 \quad 1 \quad 0 \quad 0 \quad 0) - 8$$

s.t.

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$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = (2 \;\; 1 \;\; 0 \;\; 0 \;\; 0) - 8$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \end{pmatrix} - 8$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

$B = \{1, 4\}$ is an optimal basis with the basic solution $(4, 0, 0, 0, 0)^\top$.

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct the auxiliary problem.

$$\max \quad z = (2 \quad 1 \quad 0 \quad 0 \quad 0) - 8$$

s.t.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

$B = \{1, 4\}$ is an optimal basis with the basic solution $(4, 0, 0, 0, 0)^\top$.

$z = 0$ implies that $(4, 0, 0)^\top$ is a feasible solution for $(\star)$.

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Remark

$(4,0,0)^\top$ is a basic solution.

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Remark

$(4,0,0)^\top$ is a basic solution.

Exercise

Show that this will always be the case!

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}}_{A} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}}_{A} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Question

For what basis, B, is $x = (4,0,0)^\top$ a basic solution?

Phase 1. $(4,0,0)^\top$ is a feasible solution for

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Phase 2. Find an optimal solution/detect LP unbounded.

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$B = \{1,2\}$ is a feasible basis (from Phase 1).

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We can now solve the problem using Simplex, starting from B.

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➡

$x = (0,8,12)^\top$ is an optimal solution.

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$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}$$

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(Recall that Bland's rule ensures that Simplex terminates.)

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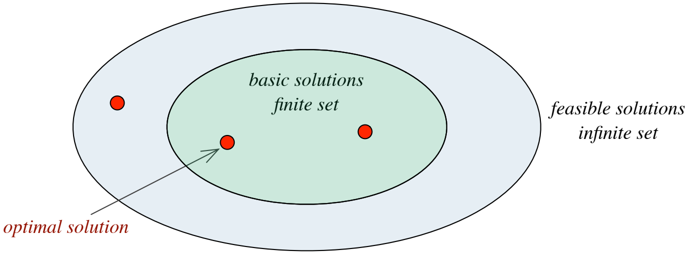

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Apply the previous theorem.

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gamegfx/iStock/Thinkstock

Our implementation