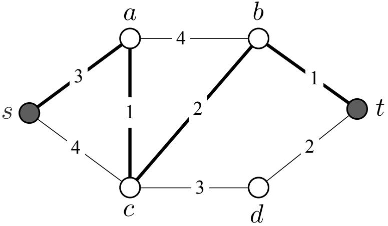
# Module 3: Duality through examples

# Recap: Shortest Paths

In an instance of the shortest path problem, we are given

- a graph $G = (V, E)$, a non-negative length $c_e$ for each edge $e \in E$, and
- a pair of vertices $s$ and $t$ in $V$.



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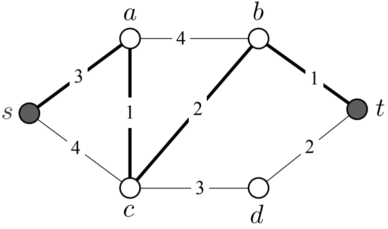

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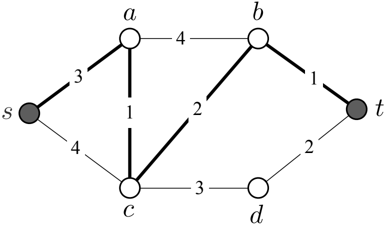
Recall: an $s, t$-path is a sequence

$$P := u_1 u_2, u_2 u_3, \ldots, u_{k-1} u_k$$

where

- $u_i u_{i+1} \in E$ for all $i$, and
- $u_1 = s$, $u_k = t$, and $u_i \neq u_j$ for all $i \neq j$.



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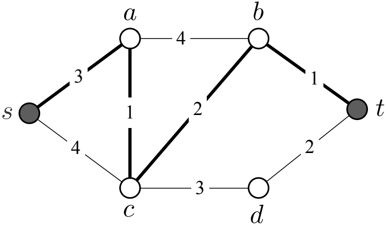


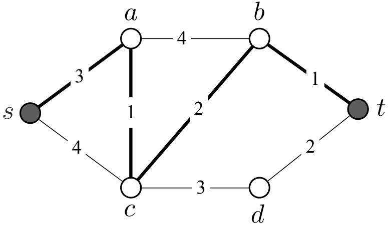

Its length is given by

$$c(P) = c_{u_1u_2} + c_{u_2u_3} + \ldots + c_{u_{k-1}u_k}$$

In the example, we see by inspection that

$$P = sa, ac, cb, bt$$

is a shortest path and that its length is $9$.



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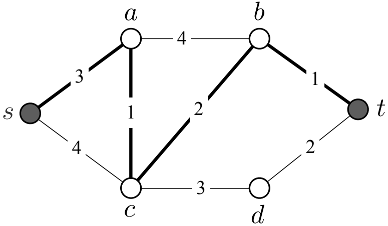


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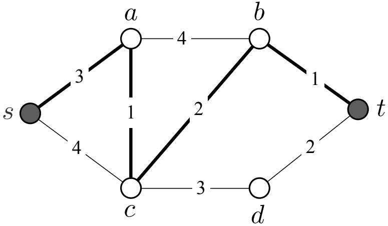


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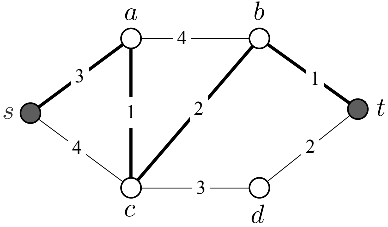



## Question

1. Given a shortest-path instance and a candidate shortest $s, t$-path $P$, is there a short proof of its optimality?

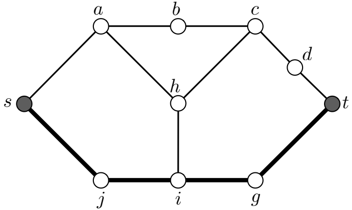
2. How can we find a shortest $s, t$-path?

We will answer both questions in this module. This lecture focus on question 1.

# Shortest Paths: Finding an Intuitive Lower Bound

# Cardinality Case

To make our lives easier, we will first consider the cardinality special case of the shortest path problem.



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We consider shortest path instances where...

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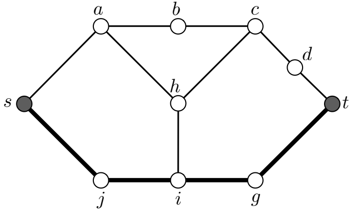

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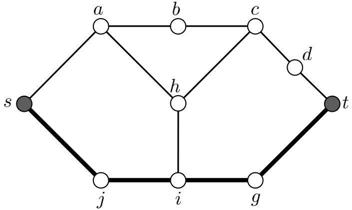

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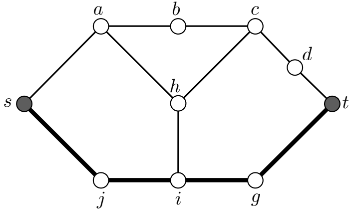


Example: In the diagram above, one easily sees that

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is a shortest $s,t$-path.

How can we prove this fact?

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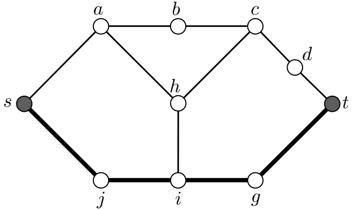


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How can we prove this fact?
$\longrightarrow$ The answer lies in s,t-cuts!

# $s, t$-cuts

**Definition**

For $U \subseteq V$, we define

$$\delta(U) = \{uv \in E \,:\, u \in U, v \notin U\}$$

and call it an $s, t$-cut if $s \in U$, and $t \notin U$.

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## Example

Let $U = \{s, a, j\}$. It follows that

$$\delta(U) = \{ab, ah, ji\}$$

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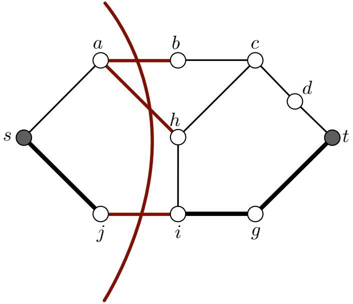

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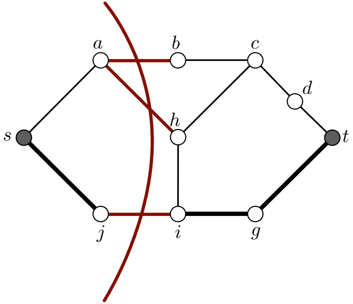


Recall:

- If $P$ is an $s, t$-path and $\delta(U)$ an $s, t$-cut, then $P$ contains an edge of $\delta(U)$.

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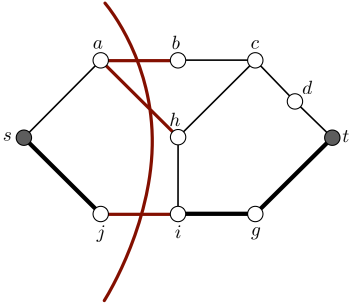
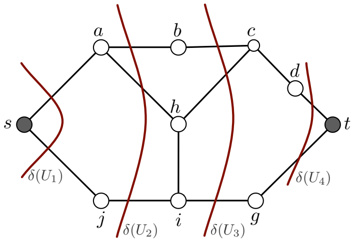



Recall:

- If $P$ is an $s, t$-path and $\delta(U)$ an $s, t$-cut, then $P$ contains an edge of $\delta(U)$.
- If $S \subseteq E$ contains an edge from every $s, t$-cut, then $S$ contains an $s, t$-path.

# From Cuts to Lower-Bounds

The example on the right shows $4$ $s,t$-cuts, $\delta(U_1), \delta(U_2), \delta(U_3), \delta(U_4)$.

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\begin{aligned}
\delta(U_1) &= \{sa, sj\} \\
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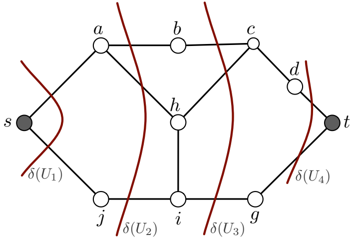
Two important notes:

(1) $\delta(U_i) \cap \delta(U_j) = \emptyset$ for $i \neq j$ and

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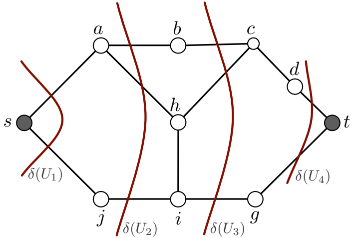


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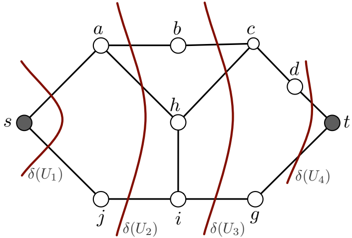
$\longrightarrow$ Every $s,t$-path must have at least $4$ edges.

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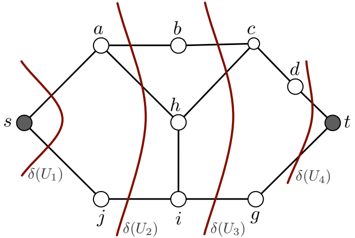


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$\longrightarrow$ Every $s,t$-path must have at least $4$ edges.
$\longrightarrow$ $sj, ji, ig, gt$ is a shortest $s,t$-path!

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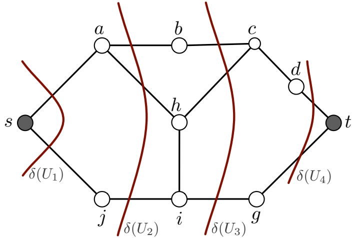
# From Cuts to Lower-Bounds

**Question**

Notice: $hi$ is not in any of the $\delta(U_i)$. Does this mean that $hi$ is not on any shortest $s,t$-path?

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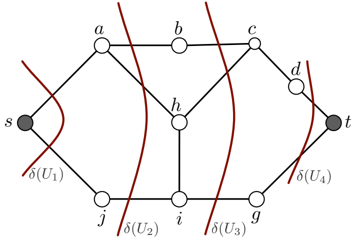


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Notice: $hi$ is not in any of the $\delta(U_i)$. Does this mean that $hi$ is not on any shortest $s,t$-path?

Yes!

An $s,t$-path that contains $hi$ must also contain an edge from each of the $s,t$-cuts $\delta(U_i)$.

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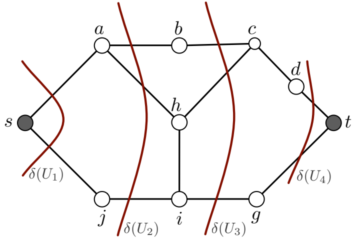


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An $s,t$-path that contains $hi$ must also contain an edge from each of the $s,t$-cuts $\delta(U_i)$. $\longrightarrow$ It must contain at least $5$ edges!

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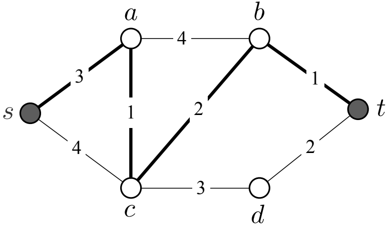
In general instances, we assign a non-negative width $y_U$ to every $s,t$-cut $\delta(U)$.



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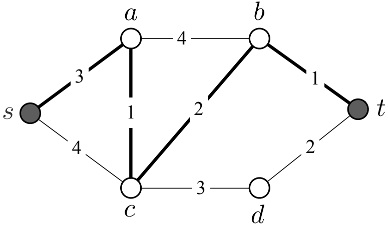


## Definition

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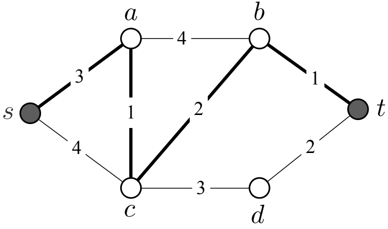


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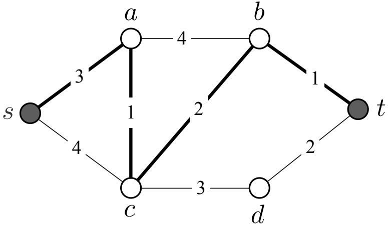


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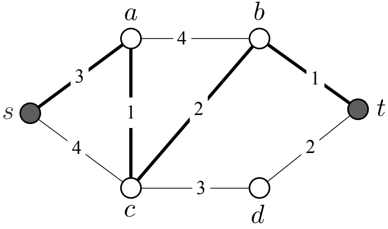


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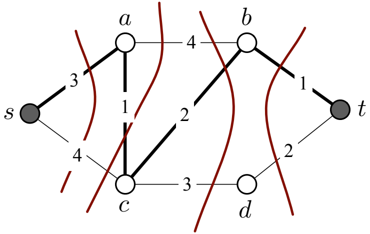
Using math: $y$ is feasible if for all $e$

$$\sum (y_U : \delta(U)\ s, t\text{-cut and } e \in E) \leq c_e$$

# Back to the General Case

Consider the example on the right with $4$ $s,t$-cuts.

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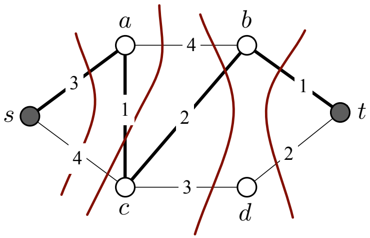
Consider the example on the right with $4$ $s, t$-cuts.

The width assignment

$$\begin{aligned} y_{U_1} &= 3 \\ y_{U_2} &= 1 \\ y_{U_3} &= 2 \\ y_{U_4} &= 1 \end{aligned}$$

is easily checked to be feasible.

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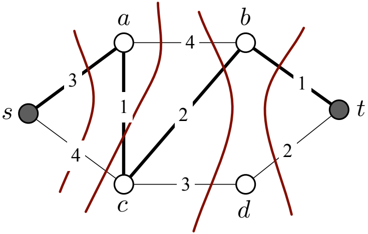
**Proposition**

If $y$ is a feasible width assignment, then any $s,t$-path must have length at least

$$\sum(y_U \,:\, U\ s,t\text{-cut}).$$

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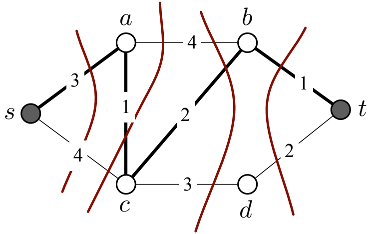


Example:

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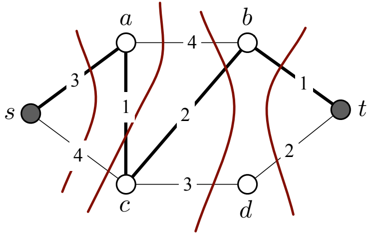


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$$y_{U_1} + y_{U_2} + y_{U_3} + y_{U_4} = 7$$

$\longrightarrow$ Path $sa, ac, cb, bt$ is a shortest path!

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$$\sum(y_U \,:\, U \;\, s,t\text{-cut}).$$

Example:

$$y_{U_1} + y_{U_2} + y_{U_3} + y_{U_4} = 7$$

$\longrightarrow$ Path $sa, ac, cb, bt$ is a shortest path!

**Proof:** Consider an $s,t$-path $P$. It follows that

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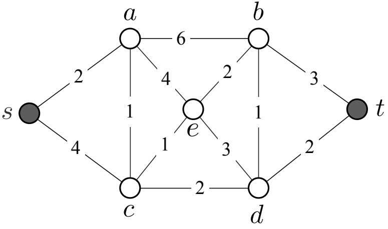
# One More Example

Question: Can you spot a shortest $s, t$-path?



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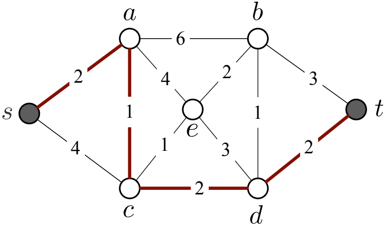

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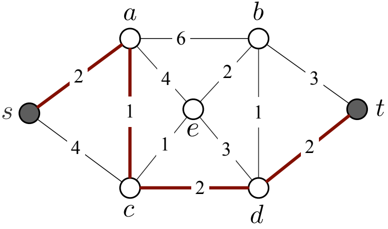

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Question: Can you spot a shortest $s, t$-path?
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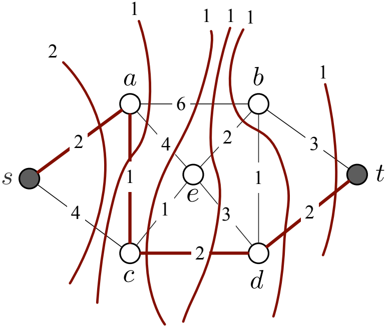
Question: Can you prove your guess?
$\longrightarrow$ Yes! There is a feasible dual width assignment of value 7:

$$\begin{aligned} y_{\{s\}} &= 2 \\ y_{\{s,a\}} &= 1 \\ y_{\{s,a,c\}} &= 1 \\ y_{\{s,a,c,e\}} &= 1 \\ y_{\{s,a,c,d,e\}} &= 1 \\ y_{\{s,a,b,c,d,e\}} &= 1 \end{aligned}$$



# One More Example

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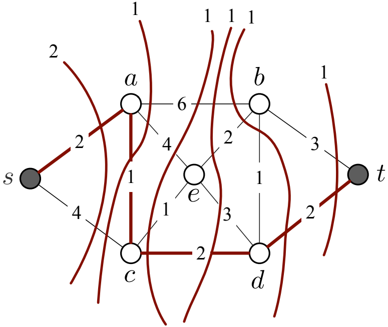
(A) In an instance with a shortest path, can we always find feasible widths to prove optimality?

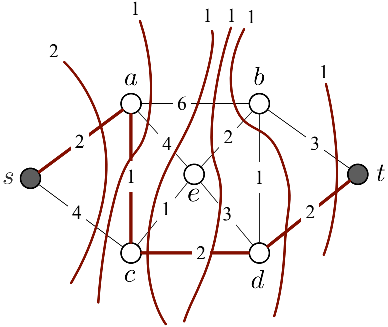


# One More Example

**Question**

(A) In an instance with a shortest path, can we always find feasible widths to prove optimality?

(B) If so, how do we find a path and these widths?



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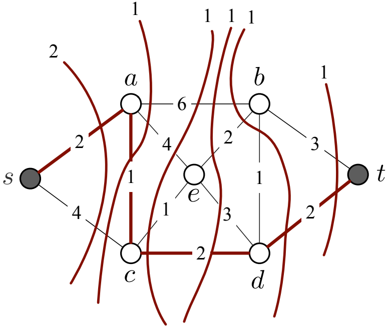


We will answer (A) affirmatively, and provide an efficient algorithm for (B) shortly.

# Recap

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- If $y$ is a feasible width assignment and $P$ an $s, t$-path, then

$$c(P) \geq \sum y_U$$