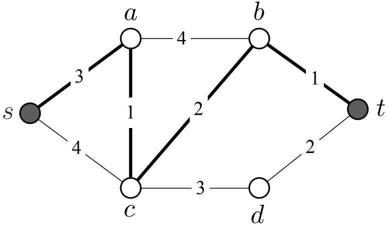
Module 3: Duality through examples (Weak Duality)

Recap: Feasible Widths

Suppose we are given an instance of the shortest path problem...



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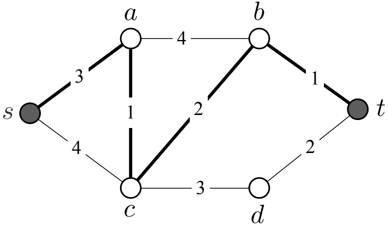
Suppose we are given an instance of the shortest path problem…

- a graph $G = (V, E)$,
- a non-negative length c_e for each edge $e \in E$, and
- a pair of vertices s and t in V.



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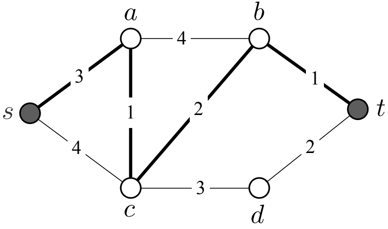

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$$\{y_U \,:\, \delta(U)\ s,t\text{-cut}\}$$

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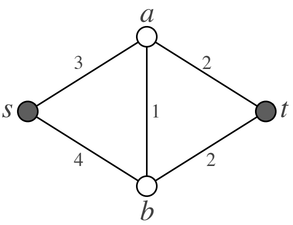

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and this is feasible if the total width of cuts containing edge e is no more than c_e, for all $e \in E$.

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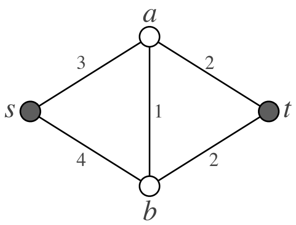

Proposition

If y is a feasible width assignment, then any s,t-path must have length at least

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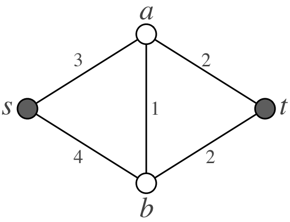

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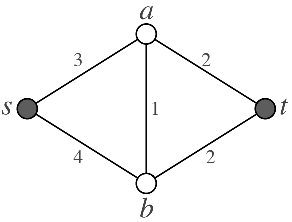

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but, as we will now see, there is a constructive and quite mechanical way to derive the Proposition via linear programming!

An Instructive Example LP

The LP on the right is feasible…

$$\begin{aligned} \min \quad & (2,3)x \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

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Feasible widths provide a lower-bound on the length of a shortest s, t-path…

Question

Can we find a good lower-bound on the objective value of the above LP?

Deriving Valid Inequalities

Let's suppose that x is feasible for the LP on the right.

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and it also satisfies

$$\begin{aligned} (2,1)x &\geq 20 \\ +\ (1,1)x &\geq 18 \\ +\ (-1,1)x &\geq 8 \\ \hline =\ (2,3)x &\geq 46 \end{aligned}$$

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Additionally, it satisfies

$$\begin{array}{r} y_1 \cdot (2,1)x \geq y_1 \cdot 20 \\ +\ y_2 \cdot (1,1)x \geq y_2 \cdot 18 \\ +\ y_3 \cdot (-1,1)x \geq y_3 \cdot 8 \\ \hline =\ (2y_1 + y_2 - y_3, y_1 + y_2 + y_3)x \\ \geq 20y_1 + 18y_2 + 8y_3 \end{array}$$

for $y_1, y_2, y_3 \geq 0$.

So, if x is feasible for the LP on the right, it also satisfies

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq$$

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E.g., for $y = (0, 2, 1)^\top$, we obtain

$$(1, 3)x \geq 44$$

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Therefore,

$$z(x) = (2, 3)x$$

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Therefore,

$$\begin{aligned} z(x) &= (2,3)x \\ &\geq (2,3)x + 44 - (1,3)x \\ &= 44 + (1,0)x \end{aligned}$$

Since $x \geq \mathbb{0}$, it follows that

$$z(x) \geq 44$$

for every feasible solution x!

State of Affairs

We now know that

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We now know that

(i) $x^2 = (5,13)^\top$ is a solution to the LP of value 49 and

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Can we find a better lowerbound on $z(x)$ for a feasible x?

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Lowerbounding $z(x)$ Systematically!

We know that a feasible x satisfies

$$0 \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} - (y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x$$

for any $y_1, y_2, y_3 \geq 0$.

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for any $y_1, y_2, y_3 \geq 0$. Therefore,

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} + \left((2,3) - (y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \right) x \quad (\star)$$

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We want the second term to be non-negative. Since $x \geq \mathbb{0}$, this amounts to choosing y such that

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \leq (2, 3)$$

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With such a y we then have from $(\star)$:

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix}$$

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So, we choose $y \geq \mathbb{0}$ such that

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Find the best possible lower-bound on z.

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Find the best possible lower-bound on z. I.e., find $y \geq \mathbb{0}$ such that $(\star)$ holds, and the right-hand side of $(\Diamond)$ is maximized!

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Solving it gives:

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and the objective value is 49.

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Since $x^2 = (5, 13)^\top$ is a feasible solution with value 49, it must be optimal!

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Therefore,

$$\begin{aligned} z(x) &= c^\top x \\ &\geq c^\top x + y^\top b - y^\top Ax \end{aligned}$$

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Suppose now we are given the LP

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The best lower-bound on $z(x)$ can be found by the following LP:

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The Dual LP

The linear program is called the dual of primal LP

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Lowerbounding the Length of s, t-Paths

Recap: Shortest Path LP

Given a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, the shortest-path LP is

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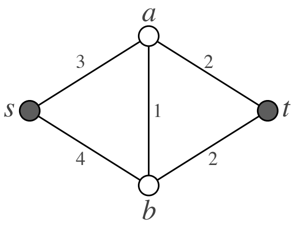
Let's look at an example!

Shortest Path: Example

On the right, we see a sample instance of the shortest-path problem.



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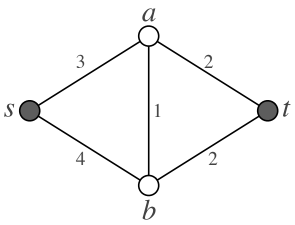

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Here is the corresponding IP:

$$\min \quad (3, 4, 1, 2, 2)x$$

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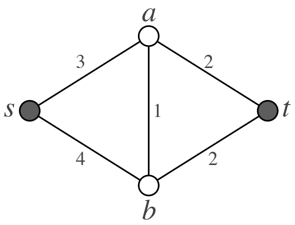

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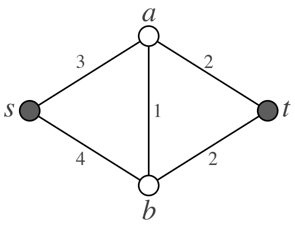

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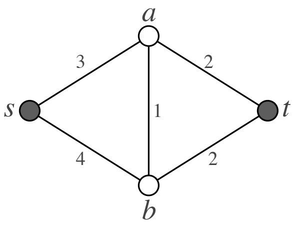

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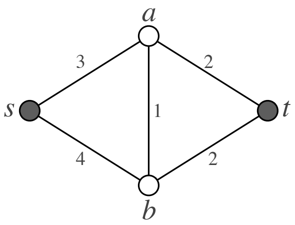

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is feasible for the IP, and its value is 6.

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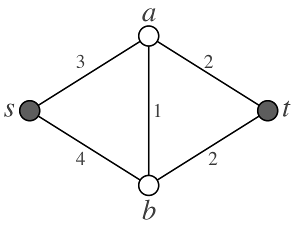

Example:

$$P = sa, ab, bt$$

is an s,t-path.

$$x = (1,0,1,0,1)^T$$

is feasible for the IP, and its value is 6.

Remark

The optimal value of the shortest path IP is, at most, the length of a shortest s,t-path.

$$\min \quad (3,4,1,2,2)x \qquad \text{(P)}$$

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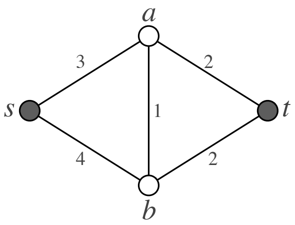

Note that dropping the integrality restriction can not increase the optimal value.

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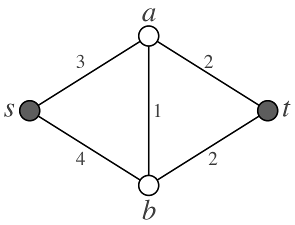

Note that dropping the integrality restriction can not increase the optimal value.

The resulting LP is called the linear programming relaxation of the IP.

$$
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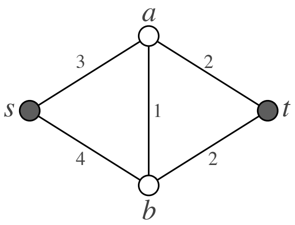

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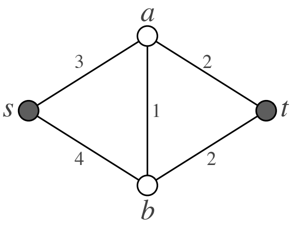
Straight from Weak Duality theorem, we have that:

Remark

The dual of (P) has optimal value no larger than that of (P)!

The dual of the shortest path LP on the previous slide is given by



$$\max \quad \mathbb{1}^\top y$$

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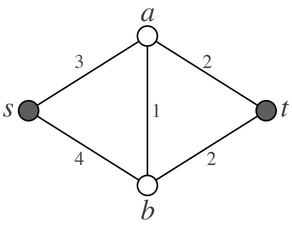

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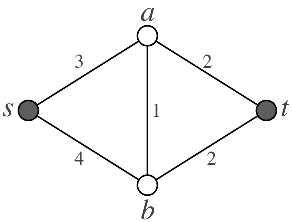

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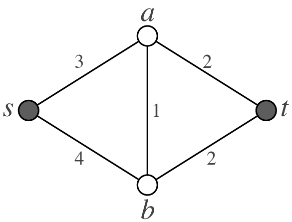

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Focus on the constraint for edge ab:

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The left-hand side is precisely the y-value assigned to s,t-cuts containing ab!

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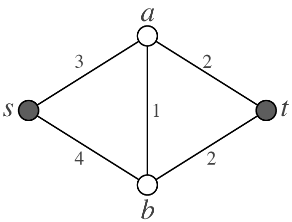

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Remark

y is feasible for the above LP if and only if it is a feasible width assignment for the s,t-cuts in the given shortest path instance!

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- The LP relaxation of an integer program is obtained by dropping the integrality restriction.
- The dual of the shortest path LP is given by

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