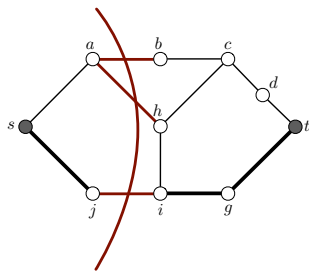


Module 4: Duality Theory (Weak Duality)

Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible 0,1-solutions for the LP

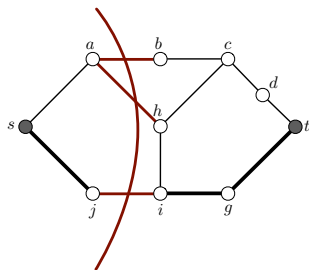
$$\begin{aligned} \min \quad & \sum (c_e x_e : e \in E) \\ \text{s.t.} \quad & \sum (x_e : e \in \delta(U)) \geq 1 \\ & (U \subseteq V, s \in U, t \notin U) \\ & x \geq 0 \end{aligned}$$



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$$\min\{c^T x : Ax \geq b, x \geq 0\}$$

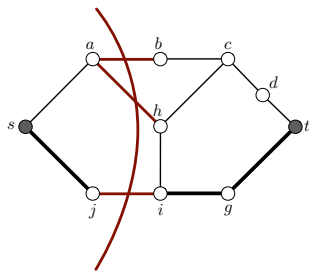
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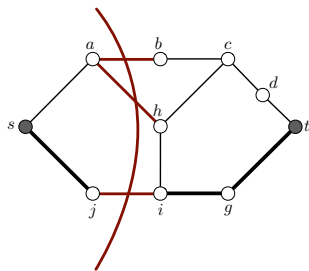
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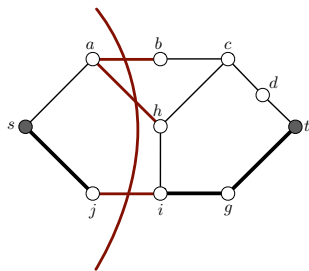
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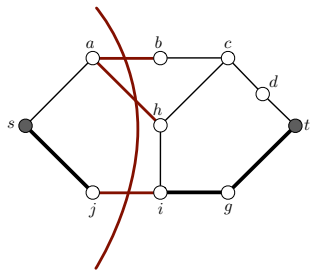


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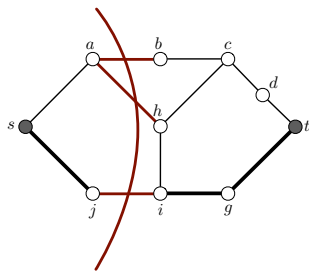


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The **dual** of (P) is given by

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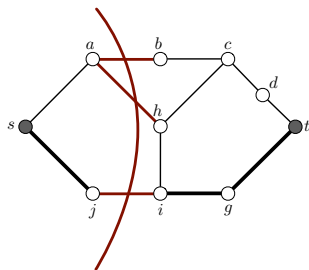
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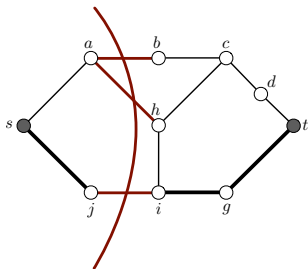
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Theorem

If \bar{x} is feasible for (P) and \bar{y} is feasible for (D), then $b^T \bar{y} \leq c^T \bar{x}$.

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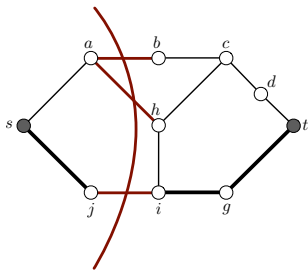
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Equivalent: y feasible widths and P an s, t -path $\rightarrow \mathbb{1}^T y \leq c(P)$

This Lecture

Question: Can we find lower-bounds on the optimal value of a **general** LP?

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Weak Duality in General

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A: As before:

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$$\max (1, 0, 2)x \quad (P)$$

$$\text{s.t. } \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

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Substitute:

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This is **consistent** with the earlier discussion we had!

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Example 3:

$$\max (12, 26, 20)x \quad (P)$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

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(P _{max})			(P _{min})		
max	$c^T x$	\leq constraint	≥ 0 variable	min	$b^T y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \leq b$	\geq constraint	≤ 0 variable		$A^T y \leq c$
	$x \geq 0$	≥ 0 variable	\geq constraint		$y \geq 0$
		free variable	$=$ constraint		
		≤ 0 variable	\leq constraint		

Example 3:

$$\max (12, 26, 20)x \quad (P)$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min (-2, 2, 13)y \quad (D)$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

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Feasible solutions: $\bar{x} = (5, -3, 0)^T$ and $\bar{y} = (0, 4, -2)^T$.

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Since $(12, 26, 20)\bar{x} = (-2, 2, 13)\bar{y} = -18 \rightarrow$ **both are optimal!**

Proving the General Weak Duality Theorem

(P_{\max})		(P_{\min})	
max	$c^T x$	\leq constraint	≥ 0 variable
subject to		$=$ constraint	free variable
	$Ax \ ? \ b$	\geq constraint	≤ 0 variable
	$x \ ? \ 0$	≥ 0 variable	\geq constraint
		free variable	$=$ constraint
		≤ 0 variable	\leq constraint
			min
			subject to
			$b^T y$
			$A^T y \ ? \ c$
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General Primal LP:

$$\max c^T x$$

$$\text{s.t. } \text{row}_i(A)x \leq b_i \quad (i \in R_1)$$

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$$x_j \geq 0 \quad (j \in C_1)$$

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 \end{aligned}$$

Its **dual** according to the table:

$$\begin{aligned}
 & \min b^T y \\
 & \text{s.t. } \text{col}_j(A)^T y \geq c_j \quad (j \in C_1) \\
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We can rewrite the above LPs using **slack variables**!

Proving the General Weak Duality Theorem

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$$\bar{y}^T b = \bar{y}^T (A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s}$$

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We can show that $\bar{w}^T \bar{x} \leq 0$ and $\bar{y}^T \bar{s} \geq 0$

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We can show that $\bar{w}^T \bar{x} \leq 0$ and $\bar{y}^T \bar{s} \geq 0 \implies \bar{y}^T b \geq c^T \bar{x}$

Consequences of Weak Duality

Theorem

Let (P_{\max}) and (P_{\min}) represent the above table. If \bar{x} and \bar{y} are feasible for the two LPs, then

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Proof: (i) Suppose, for a contradiction, that \bar{y} is feasible for (P_{\min}) . By **weak duality** $\rightarrow c^T \bar{x} \leq b^T \bar{y}$ for all \bar{x} feasible for (P_{\max}) , and hence the latter is bounded.

(ii) Similar to (i)

(iii) **weak duality** \rightarrow both (P_{\max}) and (P_{\min}) bounded

Fundamental Theorem of LP \rightarrow Both LPs must have an optimal solution!



(P_{\max})			(P_{\min})		
max	$c^T x$	\leq constraint	≥ 0 variable	min	$b^T y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \leq b$	\geq constraint	≤ 0 variable		$A^T y \leq c$
	$x \geq 0$	≥ 0 variable	\geq constraint		$y \geq 0$
		free variable	$=$ constraint		
		≤ 0 variable	\leq constraint		

Recap

- We can use the above table to compute duals of **general LPs**

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	$x \geq 0$	≥ 0 variable	\geq constraint		$y \geq 0$
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Recap

- We can use the above table to compute duals of **general LPs**
- **Weak duality theorem:** if \bar{x} and \bar{y} are feasible for (P_{\max}) and (P_{\min}) , then:

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Both are **optimal** if equality holds!