Module 4: Duality Theory (Strong Duality)

Recap: Weak Duality

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \; ? \; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	free variable	$=$ constraint	$y \; ? \; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Last lecture: we described a method to construct the dual of a general linear program.

Recap: Weak Duality

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \mathrel{?} b$	≥ 0 variable	$\geq$ constraint	$A^\top y \mathrel{?} c$
$x \mathrel{?} \mathbb{0}$	free variable	$=$ constraint	$y \mathrel{?} \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right

$$
\begin{aligned}
&\max\ (2,-1,3)x && \text{(P)}\\
&\text{s.t.}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \\
&x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ free}
\end{aligned}
$$

Recap: Weak Duality

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
	$\le$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\ge$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\ge$ constraint	$A^\top y \,?\, c$
$x \,?\, 0$	free variable	$=$ constraint	$y \,?\, 0$
	≤ 0 variable	$\le$ constraint	

Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right – a max LP that falls in the left $(\mathrm{P}_{\max})$ part of the table.

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.}\quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \le \\ = \\ \ge \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \ge 0,\ x_2 \le 0,\ x_3 \text{ free}$$

Recap: Weak Duality

$(P_{\max})$				$(P_{\min})$	
max $c^\top x$ subject to $Ax \,?\, b$, $x \,?\, \mathbb{0}$		$\le$ constraint	≥ 0 variable		min $b^\top y$ subject to $A^\top y \,?\, c$, $y \,?\, \mathbb{0}$
		$=$ constraint	free variable		
		$\ge$ constraint	≤ 0 variable		
		≥ 0 variable	$\ge$ constraint		
		free variable	$=$ constraint		
		≤ 0 variable	$\le$ constraint		

Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right – a max LP that falls in the left $(P_{\max})$ part of the table.

$\longrightarrow$ The dual of (P) is a min LP.

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \le \\ = \\ \ge \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \ge 0, x_2 \le 0, x_3 \text{ free}$$

Recap: Weak Duality

(P_{max})				
max $c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} 0$	$\leq$ constraint $=$ constraint $\geq$ constraint ≥ 0 variable free variable ≤ 0 variable	≥ 0 variable free variable ≤ 0 variable $\geq$ constraint $=$ constraint $\leq$ constraint	min $b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} 0$	

The right half of the header row is (P_{min}).

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \mathbin{?} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$y \mathbin{?} 0$$

Recap: Weak Duality

$(\mathrm{P}_{\max})$				
max $c^\top x$ subject to $Ax \mathbin{?} b$, $x \mathbin{?} 0$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbin{?} c$, $y \mathbin{?} 0$	$(\mathrm{P}_{\min})$
	$=$ constraint	free variable		
	$\geq$ constraint	≤ 0 variable		
	≥ 0 variable	$\geq$ constraint		
	free variable	$=$ constraint		
	≤ 0 variable	$\leq$ constraint		

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \ ?\ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0$$

Recap: Weak Duality

$(\mathrm{P}_{\max})$				$(\mathrm{P}_{\min})$	
max $c^\top x$ subject to $Ax \; ? \; b$, $x \; ? \; 0$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \; ? \; c$, $y \; ? \; 0$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

$$\max \ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \ \begin{matrix} \leq \\ = \\ \geq \end{matrix} \ \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

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$$y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0$$

Recap: Weak Duality

$$\begin{array}{ll} \max\ (2,-1,3)x & \text{(P)} \\ \text{s.t.}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{array}{c} \le \\ = \\ \ge \end{array} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} & \\ x_1 \ge 0, x_2 \le 0, x_3 \text{ free} & \end{array}$$

$$\begin{array}{ll} \min\ (2,1,-2)y & \text{(D)} \\ \text{s.t.}\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \begin{array}{c} \ge \\ \le \\ = \end{array} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} & \\ y_1 \ge 0, y_2 \text{ free}, y_3 \le 0 & \end{array}$$

Weak Duality Theorem

if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D),

$$\implies \quad c^T\bar{x} \le b^T\bar{y}$$

Recap: Weak Duality

$$\begin{aligned} &\max\ (2,-1,3)x && \text{(P)} \\ &\text{s.t.}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \\ &\quad x_1 \geq 0, x_2 \leq 0, x_3 \text{ free} \end{aligned}$$

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Weak Duality Theorem

if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D),

$$\implies \quad c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then both $\bar{x}$ and $\bar{y}$ are optimal.

This Lecture: Strong Duality

(P_{max})			(P_{min})	
max $c^\top x$ subject to $Ax \; ? \; b$ $x \; ? \; \mathbb{0}$	$\leq$ constraint $=$ constraint $\geq$ constraint ≥ 0 variable free variable ≤ 0 variable	≥ 0 variable free variable ≤ 0 variable $\geq$ constraint $=$ constraint $\leq$ constraint	min $b^\top y$ subject to $A^\top y \; ? \; c$ $y \; ? \; \mathbb{0}$	

Question

Can we always find feasible solutions $\bar{x}$ and $\bar{y}$ to a primal-dual pair, (P_{max}), (P_{min}), such that $c^T\bar{x} = b^T\bar{y}$?

This Lecture: Strong Duality

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \mathbin{?} b$	≥ 0 variable	$\geq$ constraint	$A^\top y \mathbin{?} c$
$x \mathbin{?} \mathbb{0}$	free variable	$=$ constraint	$y \mathbin{?} \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Question

Can we always find feasible solutions $\bar{x}$ and $\bar{y}$ to a primal-dual pair, $(\mathrm{P}_{\max})$, $(\mathrm{P}_{\min})$, such that $c^T\bar{x} = b^T\bar{y}$?

Strong Duality Theorem

If $(\mathrm{P}_{\max})$ has an optimal solution $\bar{x}$, then $(\mathrm{P}_{\min})$ has an optimal solution $\bar{y}$ such that $c^T\bar{x} = b^T\bar{y}$.

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

$$
\begin{aligned}
\max\ & c^T x & \text{(P)}\\
\text{s.t. } & Ax = b \\
& x \geq \mathbb{0}
\end{aligned}
$$

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$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

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Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

Let's assume (P) has an optimal solution.

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Let's assume (P) has an optimal solution.

$\longrightarrow$ 2-Phase Simplex terminates with an optimal basis B

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Let's assume (P) has an optimal solution.

$\longrightarrow$ 2-Phase Simplex terminates with an optimal basis B (Why?)

We can rewrite (P) for basis B:

$$\begin{aligned} \max\ & z = \bar{y}^T b + \bar{c}^T x \\ \text{s.t. } & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

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where:

$$\bar{y} = A_B^{-T} c_B$$

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and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$

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and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$

Recall that (P) and (P') are equivalent!

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

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and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$

Recall that (P) and (P') are equivalent!

$\longrightarrow$ $\bar{x}$ is feasible in (P), and has same objective value in (P) and (P')

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

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$$c^T \bar{x} \ = \qquad \bar{y}^T b + \bar{c}^T \bar{x}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

where:

$$\bar{y} = A_B^{-T} c_B$$
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Strong Duality – for LPs in SEF

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$$\begin{aligned} \max\ & c^T x \qquad \text{(P)} \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$

$$\begin{aligned} \min\ & b^T y \qquad \text{(D)} \\ \text{s.t. } & A^T y \geq c \end{aligned}$$

Recall that (P) and (P') are equivalent!

$\longrightarrow$ $\bar{x}$ is feasible in (P), and has same objective value in (P) and (P')

$$\begin{aligned} c^T \bar{x} &= \bar{y}^T b + \bar{c}^T \bar{x} \\ &= \bar{y}^T b + \bar{c}_N^T \bar{x}_N \end{aligned}$$

where:

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Strong Duality – for LPs in SEF

We can rewrite (P) for basis B:

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$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

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Goal: Show that $\bar{y}$ is dual feasible.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

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and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$ and $c^T \bar{x} = b^T \bar{y}$.

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$$x \geq \mathbb{0}$$

and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$ and $c^T \bar{x} = b^T \bar{y}$.

Note that B is an optimal basis $\longrightarrow$ $\bar{c} \leq \mathbb{0}$

$$\max\ c^T x \qquad \text{(P)}$$
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We can rewrite (P) for basis B:

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and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$ and $c^T \bar{x} = b^T \bar{y}$.

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$$\longrightarrow \qquad c^T - \bar{y}^T A \leq \mathbb{0}$$

$$\max\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

$$\min\ b^T y \qquad \text{(D)}$$
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$$\max\ z = \bar{y}^T b + \bar{c}^T x \qquad \text{(P')}$$
$$\text{s.t. } x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$
$$x \geq \mathbb{0}$$

and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$ and $c^T \bar{x} = b^T \bar{y}$.

Note that B is an optimal basis $\longrightarrow$ $\bar{c} \leq \mathbb{0}$

$$\longrightarrow \qquad c^T - \bar{y}^T A \leq \mathbb{0}$$

Equivalently, $A^T \bar{y} \geq c$,

$$\max\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y \geq c$$

where:

$$\bar{y} = A_B^{-T} c_B$$
$$\bar{c}^T = c^T - \bar{y}^T A$$

Strong Duality – for LPs in SEF

We can rewrite (P) for basis B:

$$\max\ z = \bar{y}^T b + \bar{c}^T x \qquad \text{(P')}$$
$$\text{s.t. } x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$
$$x \geq \mathbb{0}$$

and: $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$ and $c^T \bar{x} = b^T \bar{y}$.

Note that B is an optimal basis $\longrightarrow$ $\bar{c} \leq \mathbb{0}$

$$\longrightarrow \qquad c^T - \bar{y}^T A \leq \mathbb{0}$$

Equivalently, $A^T \bar{y} \geq c$,
meaning $\bar{y}$ is dual feasible!

$$\max\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y \geq c$$

where:

$$\bar{y} = A_B^{-T} c_B$$
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Strong Duality Theorem

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Let (P) and (D) be a primal-dual pair of LPs. If (P) has an optimal solution, then (D) has one, and their objective values equal.

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Fundamental Theorem of LP $\longrightarrow$ (P) has an optimal solution.

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Note: (P) is feasible and (D) is feasible $\longrightarrow$ (P) cannot be unbounded

Fundamental Theorem of LP $\longrightarrow$ (P) has an optimal solution.

Subtly different version via previous results:

Strong Duality Theorem – Feasibility Version

Let (P) and (D) be primal-dual pair of LPs. If both are feasible, then both have optimal solutions of the same objective value.

Possible Outcomes of Primal-Dual Pair (P), (D)

(D) \ (P)	optimal solution	unbounded	infeasible
optimal solution			
unbounded			
infeasible			

Possible Outcomes of Primal-Dual Pair (P), (D)

(D) \ (P)	optimal solution	unbounded	infeasible
optimal solution	possible ①		
unbounded			possible ⑥
infeasible		possible ⑧	

- ①, ⑥, and ⑧ many examples exist

Possible Outcomes of Primal-Dual Pair (P), (D)

(D) \ (P)	optimal solution	unbounded	infeasible
optimal solution	possible ①	impossible ②	
unbounded			possible ⑥
infeasible		possible ⑧	

- ①, ⑥, and ⑧ many examples exist
- ② follows directly from Weak Duality as follows:

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infeasible		possible ⑧	

- ①, ⑥, and ⑧ many examples exist
- ② follows directly from Weak Duality as follows:

  Suppose, for a contradiction, that (D) has an optimal solution $\bar{y}$.

$$\begin{aligned} \max\ & c^T x & \text{(P)} \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \min\ & b^T y & \text{(D)} \\ \text{s.t. } & A^T y \geq c \end{aligned}$$

Possible Outcomes of Primal-Dual Pair (P), (D)

(D) \ (P)	optimal solution	unbounded	infeasible
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- ①, ⑥, and ⑧ many examples exist
- ② follows directly from Weak Duality as follows:

  Suppose, for a contradiction, that (D) has an optimal solution $\bar{y}$.
  $c^T\bar{x} \leq b^T\bar{y}$ for all feasible primal solutions $\bar{x}$ by Weak Duality

$$\begin{aligned} \max\ & c^T x && \text{(P)} \\ \text{s.t.}\ & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

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- ①, ⑥, and ⑧ many examples exist
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  $c^T\bar{x} \leq b^T\bar{y}$ for all feasible primal solutions $\bar{x}$ by Weak Duality $\longrightarrow$ (P) is bounded!

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$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y \geq c$$

Possible Outcomes of Primal-Dual Pair (P), (D)

(P) / (D)	optimal solution	unbounded	infeasible
optimal solution	possible 1	impossible 2	
unbounded	impossible 4	impossible 5	possible 6
infeasible		possible 8	

- 1, 6, and 8 many examples exist
- 2 follows directly from Weak Duality as follows:

  Suppose, for a contradiction, that (D) has an optimal solution $\bar{y}$.
  $c^T\bar{x} \leq b^T\bar{y}$ for all feasible primal solutions $\bar{x}$ by Weak Duality $\longrightarrow$ (P) is bounded!
  Similar arguments apply to 4 and 5

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

Possible Outcomes of Primal-Dual Pair (P), (D)

(D) \ (P)	optimal solution	unbounded	infeasible
optimal solution	possible ①	impossible ②	impossible ③
unbounded	impossible ④	impossible ⑤	possible ⑥
infeasible	impossible ⑦	possible ⑧	

- ①, ⑥, and ⑧ many examples exist
- ② follows directly from Weak Duality as follows:

  Suppose, for a contradiction, that (D) has an optimal solution $\bar{y}$.
  $c^T\bar{x} \leq b^T\bar{y}$ for all feasible primal solutions $\bar{x}$ by Weak Duality $\longrightarrow$ (P) is bounded!
  Similar arguments apply to ④ and ⑤
- ③, ⑦ follow directly from Strong Duality

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$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y \geq c$$

Possible Outcomes of Primal-Dual Pair (P), (D)

(D) \ (P)	optimal solution	unbounded	infeasible
optimal solution	possible ①	impossible ②	impossible ③
unbounded	impossible ④	impossible ⑤	possible ⑥
infeasible	impossible ⑦	possible ⑧	possible ⑨

- ①, ⑥, and ⑧ many examples exist
- ② follows directly from Weak Duality as follows:

  Suppose, for a contradiction, that (D) has an optimal solution $\bar{y}$.
  $c^T\bar{x} \leq b^T\bar{y}$ for all feasible primal solutions $\bar{x}$ by Weak Duality $\longrightarrow$ (P) is bounded!
  Similar arguments apply to ④ and ⑤
- ③, ⑦ follow directly from Strong Duality
- I'll leave ⑨ for you to do as an exercise!

$$\begin{aligned} \max\ & c^T x \qquad \text{(P)} \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \min\ & b^T y \qquad \text{(D)} \\ \text{s.t. } & A^T y \geq c \end{aligned}$$

Recap

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Let (P) and (D) be a primal-dual pair of LPs. If (P) has an optimal solution, then (D) has one, and their objective values equal.

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optimal solution	possible ①	impossible ②	impossible ③
unbounded	impossible ④	impossible ⑤	possible ⑥
infeasible	impossible ⑦	possible ⑧	possible ⑨