Module 5: Integer Programs (Cutting Planes)

Overview

In this lecture, we will:

Investigate a class of algorithms known as cutting planes.

Overview

In this lecture, we will:

Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

Overview

In this lecture, we will:

Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

OUR FIRST INTEGER PROGRAM:

Overview

In this lecture, we will:

Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

OUR FIRST INTEGER PROGRAM:

$$
\begin{aligned}
&\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
&\qquad x \geq \mathbf{0}, \ x \text{ integer}
\end{aligned}
$$

Overview

In this lecture, we will:

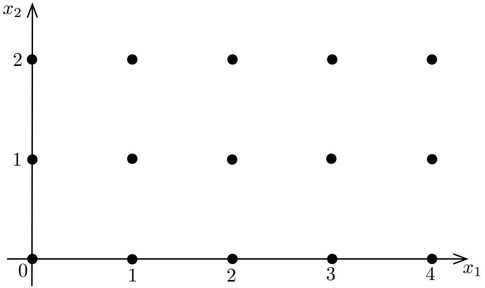
Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

Our First Integer Program:

$$\begin{aligned}
&\max \ \begin{pmatrix}2 & 5\end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix}1 & 4 \\ 1 & 1\end{pmatrix} x \leq \begin{pmatrix}8 \\ 4\end{pmatrix} \qquad \begin{matrix}(1)\\(2)\end{matrix} \\
&\qquad x \geq \mathbf{0},\ x \text{ integer}
\end{aligned}$$



Overview

In this lecture, we will:

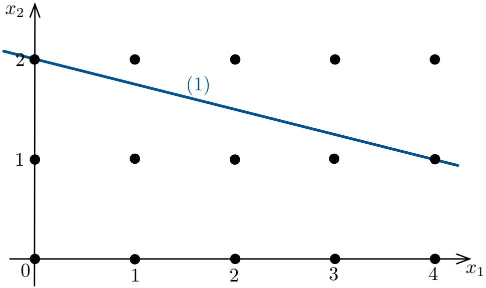
Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

Our First Integer Program:

$$\begin{aligned} &\max \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ &\qquad x \geq \mathbf{0},\ x \text{ integer} \end{aligned}$$



Overview

In this lecture, we will:

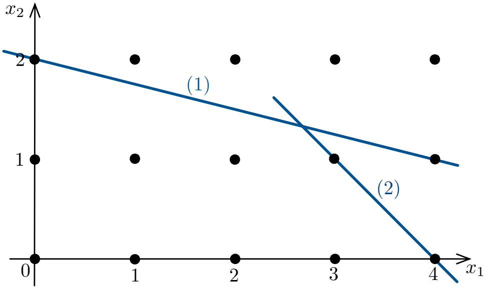
Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

Our First Integer Program:

$$\begin{aligned} &\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ &\qquad x \geq \mathbf{0}, \ x \text{ integer} \end{aligned}$$



Overview

In this lecture, we will:

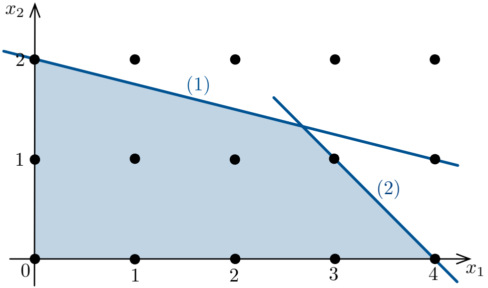
Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

Our First Integer Program:

$$\begin{aligned} &\max \; \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ &\qquad x \geq \mathbf{0}, \; x \text{ integer} \end{aligned}$$



Overview

In this lecture, we will:

Investigate a class of algorithms known as cutting planes.

Remark

We restrict ourselves to pure Integer Programs.

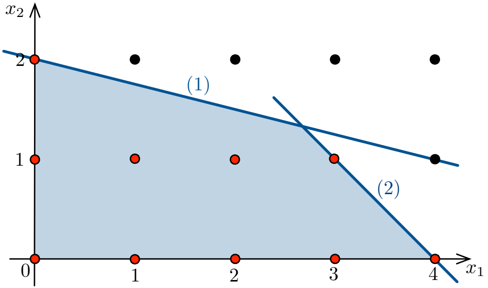
Our First Integer Program:

$$\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

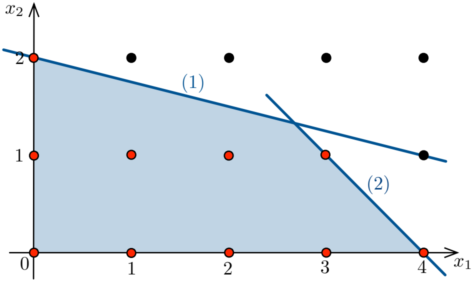
s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$x \geq \mathbf{0}, \ x \text{ integer}$$



$$\begin{aligned}
&\max\ \begin{pmatrix}2 & 5\end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix} x \le \begin{pmatrix}8\\ 4\end{pmatrix} \qquad \begin{matrix}(1)\\(2)\end{matrix} \\
&\qquad x \ge \mathbf{0},\ x \text{ integer}
\end{aligned}$$

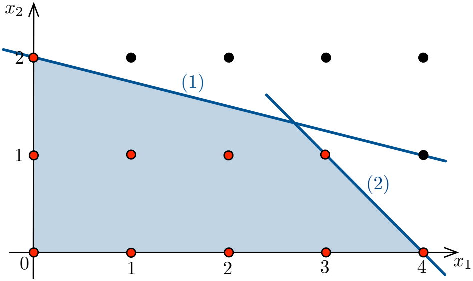


$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

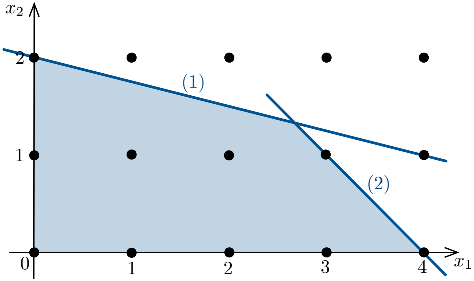
$$x \geq \mathbf{0}, \; x \text{ integer}$$



Idea

Solve the LP relaxation instead of the original IP.

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$
s. t.
$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$
$$x \geq \mathbf{0}$$



Idea

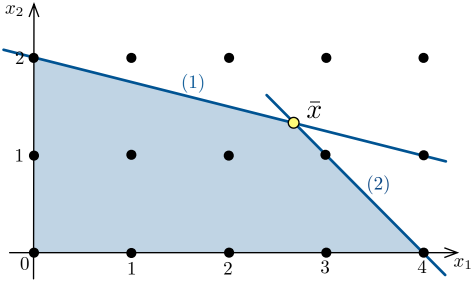
Solve the LP relaxation instead of the original IP.

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

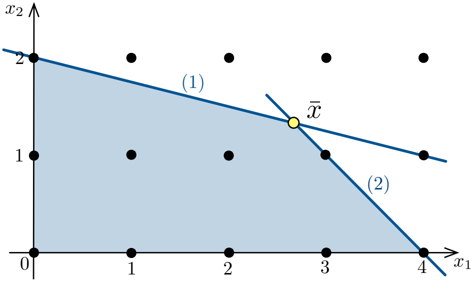
$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$x \geq \mathbf{0}$$



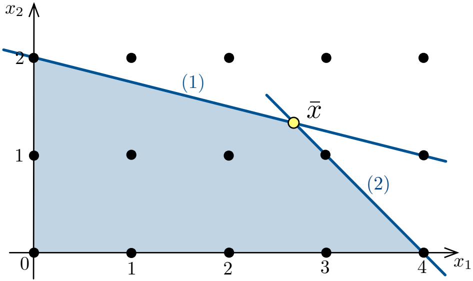
Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal.

$$\begin{array}{ll} \max & \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix} \\ & x \geq \mathbf{0} \end{array}$$



Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. NOT INTEGER!

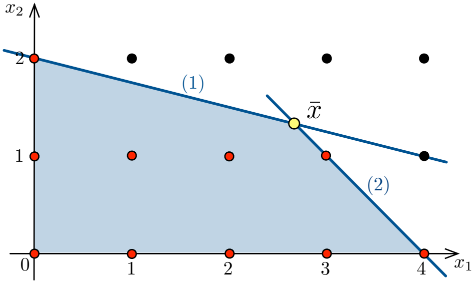
$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$
s. t.
$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$
$$x \geq \mathbf{0}$$



Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. NOT INTEGER!

We now search for a constraint $\alpha^\top x \leq \beta$ that

$$
\begin{aligned}
&\max \; \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
&\qquad x \ge \mathbf{0}
\end{aligned}
$$

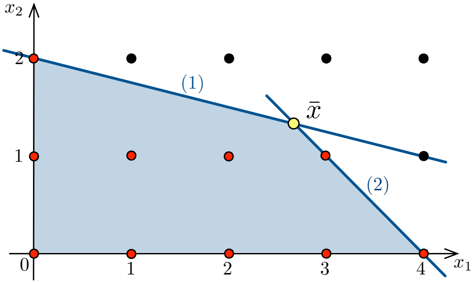


Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. Not integer!

We now search for a constraint $\alpha^\top x \le \beta$ that

- is satisfied for all feasible solutions to the IP, and

$$\begin{aligned} &\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$

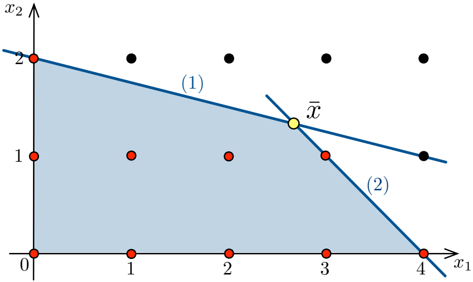


Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. NOT INTEGER!

We now search for a constraint $\alpha^\top x \leq \beta$ that

- is satisfied for all feasible solutions to the IP, and
- is not satisfied for $\bar{x}$.

$$
\begin{aligned}
\max\ & \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
\text{s. t.}\ & \\
& \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
& x \ge \mathbf{0}
\end{aligned}
$$



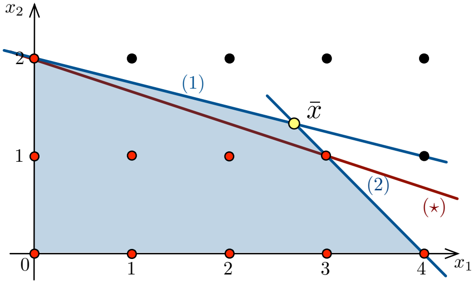
Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. NOT INTEGER!

We now search for a constraint $\alpha^\top x \le \beta$ that

- is satisfied for all feasible solutions to the IP, and
- is not satisfied for $\bar{x}$.

We will call this constraint a cutting plane for $\bar{x}$.

$$\begin{aligned} &\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$



Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. Not integer!

We now search for a constraint $\alpha^\top x \leq \beta$ that

- is satisfied for all feasible solutions to the IP, and
- is not satisfied for $\bar{x}$.

We will call this constraint a cutting plane for $\bar{x}$.

Example:

$$x_1 + 3x_2 \leq 6. \qquad (\star)$$

After adding $(\star)$ to our relaxation, we get

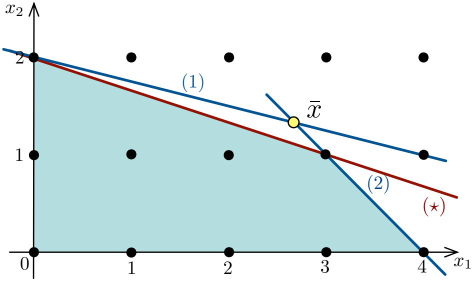
After adding $(\star)$ to our relaxation, we get

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

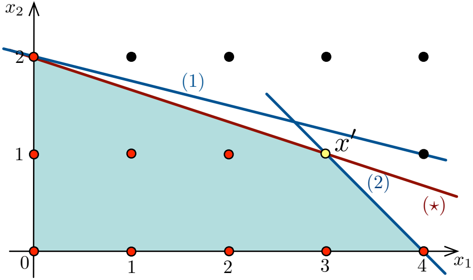
$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$x \geq \mathbf{0}$$



After adding (⋆) to our relaxation, we get

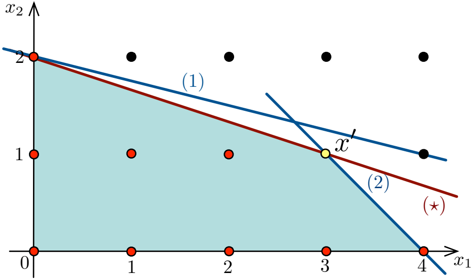
$$\begin{aligned} &\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \\ &\qquad x \ge \mathbf{0} \end{aligned}$$



Using Simplex, we get: $x' = (3, 1)^\top$ is optimal.

After adding $(\star)$ to our relaxation, we get

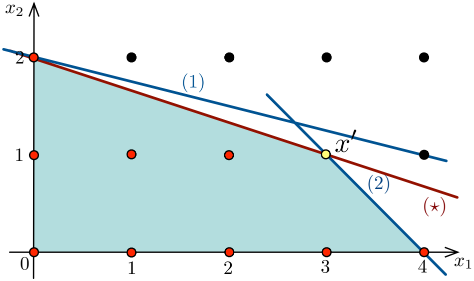
$$\begin{aligned} &\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$



Using Simplex, we get: $x' = (3, 1)^\top$ is optimal. INTEGER!

After adding ($\star$) to our relaxation, we get

$$\begin{array}{l} \max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \\ \qquad x \ge \mathbf{0} \end{array}$$

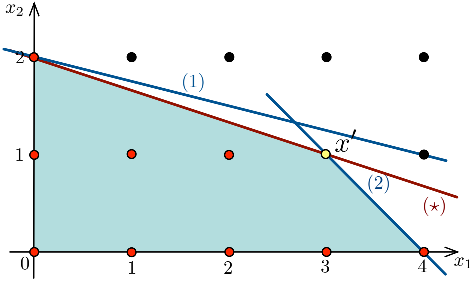


Using Simplex, we get: $x' = (3, 1)^\top$ is optimal. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

After adding $(\star)$ to our relaxation, we get

$$\begin{aligned} &\max \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$



Using Simplex, we get: $x' = (3,1)^\top$ is optimal. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

We have now solved our first IP.

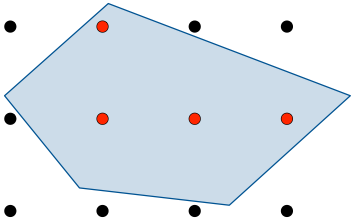
Cutting Plane Scheme

Cutting Plane Scheme

$$\max \left\{ c^\top x : Ax \leq b, x \text{ integer} \right\} \qquad \text{(IP)}$$

Cutting Plane Scheme

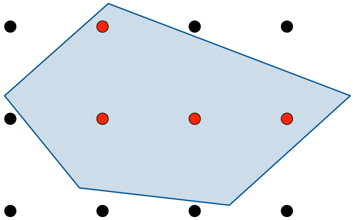
$$\max \left\{ c^\top x : Ax \le b, x \text{ integer} \right\} \quad \text{(IP)}$$



- Let (P) denote $\max\{c^\top x : Ax \le b\}$.

Cutting Plane Scheme

$$\max\left\{c^\top x : Ax \le b, x \text{ integer}\right\} \qquad \text{(IP)}$$



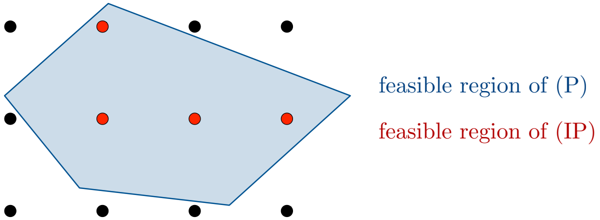
feasible region of (P)

feasible region of (IP)

- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.

Cutting Plane Scheme

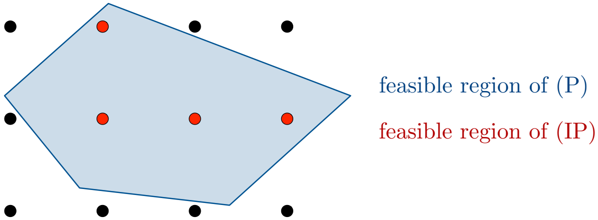
$$\max \left\{ c^\top x : Ax \le b, x \text{ integer} \right\} \qquad \text{(IP)}$$



- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.
- Let $\bar{x}$ be the optimal solution to (P).

Cutting Plane Scheme

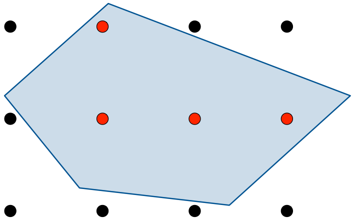
$$\max \left\{ c^\top x : Ax \leq b, x \text{ integer} \right\} \qquad \text{(IP)}$$



- Let (P) denote $\max\{c^\top x : Ax \leq b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.
- Let $\bar{x}$ be the optimal solution to (P).
- If $\bar{x}$ is integral, then STOP. $\bar{x}$ is also optimal for (IP).

Cutting Plane Scheme

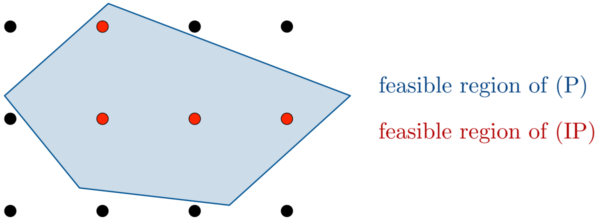
$$\max\left\{c^\top x : Ax \le b, x \text{ integer}\right\} \qquad \text{(IP)}$$



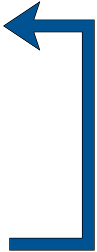
- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.
- Let $\bar{x}$ be the optimal solution to (P).
- If $\bar{x}$ is integral, then STOP. $\bar{x}$ is also optimal for (IP).
- Find a cutting plane $a^\top x \le \beta$ for $\bar{x}$.

Cutting Plane Scheme

$$\max\left\{c^\top x : Ax \le b, x \text{ integer}\right\} \qquad \text{(IP)}$$



- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.
- Let $\bar{x}$ be the optimal solution to (P).
- If $\bar{x}$ is integral, then STOP. $\bar{x}$ is also optimal for (IP).
- Find a cutting plane $a^\top x \le \beta$ for $\bar{x}$.
- Add constraint $a^\top x \le \beta$ to the system $Ax \le b$.



Question

How can we find cutting planes?

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Question

How can we find cutting planes?

Simplex does this for us!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\lfloor 3.7 \rfloor = 3$$

Question

How can we find cutting planes?

Simplex does this for us!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\lfloor 3.7 \rfloor = 3$$

$$\lfloor 62 \rfloor = 62$$

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then the floor of a, denoted $\lfloor a \rfloor$, is the largest integer $\leq a$.

Example

$$\begin{aligned} \lfloor 3.7 \rfloor &= 3 \\ \lfloor 62 \rfloor &= 62 \\ \lfloor -2.1 \rfloor &= -3 \end{aligned}$$

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$x \geq \mathbf{0}, \ x \text{ integer}$$

$$\begin{aligned} &\max \ \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ &\text{s. t.} \\ &\qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ &\qquad x \ge \mathbf{0}, \ x \text{ integer} \end{aligned}$$

Add a slack variable, $x_3 \geq 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$x \ge \mathbf{0}, \ x \text{ integer}$$

Add a slack variable, $x_3 \ge 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \ge 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

$$\max \begin{pmatrix} 2 & 5 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

$$x \ge \mathbf{0}, \ x \text{ integer}$$

Add a slack variable, $x_3 \ge 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \ge 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

Since x_1, x_2 are integers, $x_3 = 8 - x_1 - 4x_2$ and $x_4 = 4 - x_1 - x_2$ are integers.

$$\begin{array}{l} \max \begin{pmatrix} 2 & 5 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ \qquad x \geq \mathbf{0},\ x \text{ integer} \end{array}$$

Add a slack variable, $x_3 \geq 0$, and rewrite (1) as

$$x_1 + 4x_2 + x_3 = 8.$$

Add another slack variable, $x_4 \geq 0$, and rewrite (2) as

$$x_1 + x_2 + x_4 = 4.$$

Since x_1, x_2 are integers, $x_3 = 8 - x_1 - 4x_2$ and $x_4 = 4 - x_1 - x_2$ are integers.

Thus, we can rewrite the IP as

$$\begin{array}{l} \max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ \qquad x \geq \mathbf{0},\ x \text{ integer} \end{array}$$

Solving the IP

$$\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}, \; x \text{ integer}$$

Solving the IP

$$\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x$$
s. t.
$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$
$$x \geq \mathbf{0}, \ x \text{ integer}$$

We will now relax the integer program.

Solving the IP

$$\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Solving the IP

$$\max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x$$

s. t.

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

We will use the Simplex algorithm to solve this.

Solving the IP

$$\max \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x$$

s. t.

$$\begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\ 4\end{pmatrix}$$

$$x \geq \mathbf{0}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

Solving the IP

$$\begin{aligned} \max \ & \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} \ & \\ & \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{aligned} \max \ & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \ & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

Solving the IP

$$
\begin{aligned}
\max\ & \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x \\
\text{s. t.}\ & \\
& \begin{pmatrix}1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8 \\ 4\end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}
$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$
\begin{aligned}
\max\ & \begin{pmatrix}0 & 0 & -1 & -1\end{pmatrix} x + 12 \\
\text{s. t.}\ & \\
& \begin{pmatrix}1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3\end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Solving the IP

$$\begin{array}{l} \max \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} x \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$. Not integer

Solving the IP

$$\begin{aligned}&\max \begin{pmatrix}2 & 5 & 0 & 0\end{pmatrix} x\\ &\text{s. t.}\\ &\qquad \begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}8\\4\end{pmatrix}\\ &\qquad x \geq \mathbf{0}\end{aligned}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1, 2\}$ and rewrite in canonical form for B:

$$\begin{aligned}&\max \begin{pmatrix}0 & 0 & -1 & -1\end{pmatrix} x + 12\\ &\text{s. t.}\\ &\qquad \begin{pmatrix}1 & 0 & -1/3 & 4/3\\ 0 & 1 & 1/3 & -1/3\end{pmatrix} x = \begin{pmatrix}8/3\\4/3\end{pmatrix}\\ &\qquad x \geq \mathbf{0}\end{aligned}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$. Not integer

Let us use the canonical form to get a cutting plane for $\bar{x}$.

$$\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12$$

s. t.

$$\begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 = \frac{8}{3}$$

$$
\begin{array}{l}
\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12$$

s. t.

$$\begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$
\begin{array}{l}
\max \; \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
\qquad x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

$$\begin{array}{l} \max \ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

For every feasible solution to the IP, $x_1 - x_3 + x_4$ is integer.

$$\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

For every feasible solution to the IP, $x_1 - x_3 + x_4$ is integer.

Hence, every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq \left\lfloor \frac{8}{3} \right\rfloor = 2$$

$$\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \qquad (\star)$$

$$\begin{aligned}
&\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
&\qquad x \geq \mathbf{0}
\end{aligned}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \qquad (\star)$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12$$

s. t.

$$\begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \qquad (\star)$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

⇒ $(\star)$ is a cutting plane for $\bar{x}$.

$$
\begin{aligned}
&\max\ \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
&\qquad x \geq \mathbf{0}
\end{aligned}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \qquad (\star)$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

$\Rightarrow$ $(\star)$ is a cutting plane for $\bar{x}$.

We can rewrite $(\star)$ as

$$x_1 - x_3 + x_4 + x_5 = 2 \qquad \text{where} \qquad x_5 \geq 0.$$

$$\begin{array}{l} \max \begin{pmatrix} 0 & 0 & -1 & -1 \end{pmatrix} x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \tag{$\star$}$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$\underbrace{x_1}_{8/3} - \underbrace{x_3}_{0} + \underbrace{x_4}_{0} = \frac{8}{3} > 2$$

➡ $(\star)$ is a cutting plane for $\bar{x}$.

We can rewrite $(\star)$ as

$$x_1 - x_3 + x_4 + x_5 = 2 \qquad \text{where} \qquad x_5 \geq 0.$$

We now add this to the relaxation.

$$\max \quad \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12$$

s. t.

$$\begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix}$$

$$x \geq \mathbf{0}$$

$$\max \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12$$

s. t.

$$\begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Solve this using the Simplex algorithm.

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{|l|}
\hline
\max \quad \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
\qquad x \geq \mathbf{0} \\
\hline
\end{array}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{array}{|l|}
\hline
\max \quad \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} x + 11 \\
\text{s. t.} \\
\qquad \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
\qquad x \geq \mathbf{0} \\
\hline
\end{array}
$$

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix}1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}8/3 \\ 4/3 \\ 2\end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2}\end{pmatrix} x + 11 \\ \text{s. t.} & \\ & \begin{pmatrix}1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2\end{pmatrix} x = \begin{pmatrix}3 \\ 1 \\ 1\end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$.

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 0 & -1 & -1 & 0\end{pmatrix} x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2}\end{pmatrix} x + 11 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$. INTEGER!

$$
\begin{aligned}
\max \quad & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} \quad & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}
$$

Solve this using the Simplex algorithm.

Get optimal basis $B = \{1, 2, 3\}$ and rewrite in canonical form for B:

$$
\begin{aligned}
\max \quad & \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} x + 11 \\
\text{s. t.} \quad & \\
& \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{aligned}
$$

The basic optimal solution is $x' = (3, 1, 1, 0, 0)^\top$. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

$(3, 1, 1, 0, 0)^\top$ is optimal for

$$
\begin{aligned}
\max \ & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} \ & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}, \quad x \text{ integer}
\end{aligned}
$$

$(3,1,1,0,0)^\top$ is optimal for

$$
\begin{aligned}
\max \quad & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.} \quad & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0}, \quad x \text{ integer}
\end{aligned}
$$

$(3,1)^\top$ is optimal for

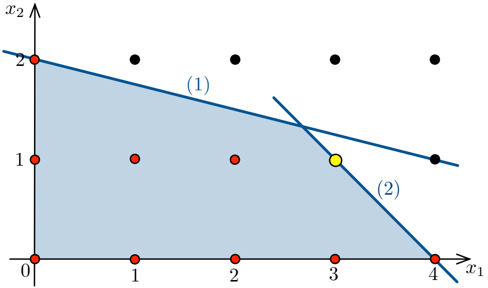
$$
\begin{aligned}
\max \quad & \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
\text{s. t.} \quad & \\
& \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
& x \geq \mathbf{0}, \ x \text{ integer}
\end{aligned}
$$

$(3,1,1,0,0)^\top$ is optimal for

$$\begin{aligned}
\max\ & \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \end{pmatrix} x + 12 \\
\text{s. t.}\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\
& x \geq \mathbf{0},\quad x \text{ integer}
\end{aligned}$$

$(3,1)^\top$ is optimal for

$$\begin{aligned}
\max\ & \begin{pmatrix} 2 & 5 \end{pmatrix} x \\
\text{s. t.}\ & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\
& x \geq \mathbf{0},\ x \text{ integer}
\end{aligned}$$



Getting Cutting Planes in General

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{aligned} \max \quad & \bar{c}^\top x + \bar{z} \\ \text{s. t.} \quad & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{aligned}$$

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$
\begin{array}{ll}
\max & \bar{c}^\top x + \bar{z} \\
\text{s. t.} & \\
& x_B + A_N x_N = b \\
& x \geq \mathbf{0}
\end{array}
$$

$$N = \{j : j \notin B\}$$

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{aligned} \max \quad & \bar{c}^\top x + \bar{z} \\ \text{s. t.} \quad & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{aligned}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{aligned} \max \quad & \bar{c}^\top x + \bar{z} \\ \text{s. t.} \quad & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{aligned}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{aligned} \max \quad & \bar{c}^\top x + \bar{z} \\ \text{s. t.} \quad & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{aligned}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER.

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{aligned} \max \quad & \bar{c}^\top x + \bar{z} \\ \text{s. t.} \quad & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{aligned}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

We know that every feasible solution to the LP relaxation satisfies

$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j = b_i.$$

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

We know that every feasible solution to the LP relaxation satisfies

$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j \leq b_i.$$

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

We know that every feasible solution to the LP relaxation satisfies

$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j \leq b_i. \quad \Longrightarrow \quad x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq b_i.$$

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable.

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to the LP relaxation satisfies

$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j \leq b_i. \quad \Longrightarrow \quad \underbrace{x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j}_{\text{integer for all } x \text{ integer}} \leq b_i.$$

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic $(\bar{x}_N = \mathbf{0}, \bar{x}_B = b)$

$r(i)$ index of i^{th} basic variable.

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to the LP relaxation satisfies

$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j \leq b_i. \quad \Longrightarrow \quad \underbrace{x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j}_{\text{integer for all } x \text{ integer}} \leq b_i.$$

Hence, every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b_i \rfloor$$

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b \rfloor \qquad (\star)$$

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b \rfloor \qquad (\star)$$

However, $\bar{x}$ does not satisfy $(\star)$ as

Solve the relaxation and get the LP in a canonical form for B.

$$
\begin{array}{|l|}
\hline
\max \quad \bar{c}^\top x + \bar{z} \\
\text{s. t.} \\
\qquad x_B + A_N x_N = b \\
\qquad x \geq \mathbf{0} \\
\hline
\end{array}
$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b \rfloor \qquad (\star)$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_{r(i)}}_{b_i} + \sum_{j \in N} \lfloor A_{ij} \rfloor \underbrace{x_j}_{=0}$$

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b \rfloor \qquad (\star)$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_{r(i)}}_{b_i} + \sum_{j \in N} \lfloor A_{ij} \rfloor \underbrace{x_j}_{=0} = b_i$$

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{|l|}\hline \max \quad \bar{c}^\top x + \bar{z} \\ \text{s. t.} \\ \qquad x_B + A_N x_N = b \\ \qquad x \geq \mathbf{0} \\ \hline \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b \rfloor \qquad (\star)$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_{r(i)}}_{b_i} + \sum_{j \in N} \lfloor A_{ij} \rfloor \underbrace{x_j}_{=0} = b_i > \lfloor b_i \rfloor .$$

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b \rfloor \qquad (\star)$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_{r(i)}}_{b_i} + \sum_{j \in N} \lfloor A_{ij} \rfloor \underbrace{x_j}_{=0} = b_i > \lfloor b_i \rfloor .$$

⟹ ($\star$) is a cutting plane for $\bar{x}$.

The Good and the Bad

The Good and the Bad

The Good News:

- The Simplex based cutting plane algorithm eventually will terminate.

The Good and the Bad

The Good News:

- The Simplex based cutting plane algorithm eventually will terminate.

The Bad News

- If implemented in this way, it will be terribly slow.

The Good and the Bad

The Good News:

- The Simplex based cutting plane algorithm eventually will terminate.

The Bad News

- If implemented in this way, it will be terribly slow.

Ways we can Improve the Algorithm

The Good and the Bad

The Good News:

- The Simplex based cutting plane algorithm eventually will terminate.

The Bad News

- If implemented in this way, it will be terribly slow.

Ways we can Improve the Algorithm

- Do not use the 2-phase Simplex to reoptimize; work with the dual.

The Good and the Bad

The Good News:

- The Simplex based cutting plane algorithm eventually will terminate.

The Bad News

- If implemented in this way, it will be terribly slow.

Ways we can Improve the Algorithm

- Do not use the 2-phase Simplex to reoptimize; work with the dual.
- Add more than one cutting plane at at time.

The Good and the Bad

The Good News:

- The Simplex based cutting plane algorithm eventually will terminate.

The Bad News

- If implemented in this way, it will be terribly slow.

Ways we can Improve the Algorithm

- Do not use the 2-phase Simplex to reoptimize; work with the dual.
- Add more than one cutting plane at at time.
- Combine it with a divide and conquer strategy (branch and bound).

Recap

Recap

- We solve the LP relaxation of an integer program and get a solution;

Recap

- We solve the LP relaxation of an integer program and get a solution;
- If the solution is integer, it is optimal for the integer program;

Recap

- We solve the LP relaxation of an integer program and get a solution;
- If the solution is integer, it is optimal for the integer program;
- Otherwise, we add a cutting plane.

Recap

- We solve the LP relaxation of an integer program and get a solution;
- If the solution is integer, it is optimal for the integer program;
- Otherwise, we add a cutting plane.
- Cutting planes can be obtained from the final canonical form.

Recap

- We solve the LP relaxation of an integer program and get a solution;
- If the solution is integer, it is optimal for the integer program;
- Otherwise, we add a cutting plane.
- Cutting planes can be obtained from the final canonical form.
- Careful implementation is key to success.