Module 6: Nonlinear Programs (Convexity)

Definition

A Nonlinear Program (NLP) is a problem of the form:

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\boxed{\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}}$$

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\boxed{\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}}$$

where

$f : \Re^n \to \Re$, and

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{|lll|} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \\ \end{array}$$

where

$f : \Re^n \to \Re$, and

$g_i : \Re^n \to \Re$ for $i = 1, \ldots, k$.

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}$$

where

$f : \Re^n \to \Re$, and

$g_i : \Re^n \to \Re$ for $i = 1, \ldots, k$.

Remark

There aren't any restrictions regarding the type of functions.

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}$$

where

$f : \Re^n \to \Re$, and

$g_i : \Re^n \to \Re$ for $i = 1, \ldots, k$.

Remark

There aren't any restrictions regarding the type of functions.

This is a very general model, but NLPs can be very hard to solve!

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \\ & g_i(x) \leq 0 \qquad (i = 1, \ldots, k) \end{array}$$

$$\begin{array}{lrcl} \min & x_2 & & \\ \text{s.t.} & & & \\ & -x_1^2 - x_2 + 2 & \leq & 0 \\ & x_2 - \frac{3}{2} & \leq & 0 \\ & x_1 - \frac{3}{2} & \leq & 0 \\ & -x_1 - 2 & \leq & 0 \end{array}$$

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \\ & g_i(x) \le 0 \qquad (i = 1, \ldots, k) \end{array}$$

$$\begin{array}{lrcl} \min & \underbrace{x_2}_{f(x)} & & \\ \text{s.t.} & & & \\ & -x_1^2 - x_2 + 2 & \le & 0 \\ & x_2 - \frac{3}{2} & \le & 0 \\ & x_1 - \frac{3}{2} & \le & 0 \\ & -x_1 - 2 & \le & 0 \end{array}$$

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$
\begin{array}{ll}
\min & f(x) \\
\text{s.t.} & \\
& g_i(x) \le 0 \qquad (i = 1, \ldots, k)
\end{array}
$$

$$
\begin{array}{lrcl}
\min & x_2 & & \\
\text{s.t.} & & & \\
& \underbrace{-x_1^2 - x_2 + 2}_{g_1(x)} & \le & 0 \\
& x_2 - \frac{3}{2} & \le & 0 \\
& x_1 - \frac{3}{2} & \le & 0 \\
& -x_1 - 2 & \le & 0
\end{array}
$$

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \\ & g_i(x) \le 0 \qquad (i = 1, \ldots, k) \end{array}$$

$$\begin{array}{lrcl} \min & x_2 & & \\ \text{s.t.} & & & \\ & -x_1^2 - x_2 + 2 & \le & 0 \\ & \underbrace{x_2 - \frac{3}{2}}_{g_2(x)} & \le & 0 \\ & x_1 - \frac{3}{2} & \le & 0 \\ & -x_1 - 2 & \le & 0 \end{array}$$

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \\ & g_i(x) \leq 0 \qquad (i = 1, \ldots, k) \end{array}$$

$$\begin{array}{lrcl} \min & x_2 & & \\ \text{s.t.} & & & \\ & -x_1^2 - x_2 + 2 & \leq & 0 \\ & x_2 - \frac{3}{2} & \leq & 0 \\ & \underbrace{x_1 - \frac{3}{2}}_{g_3(x)} & \leq & 0 \\ & -x_1 - 2 & \leq & 0 \end{array}$$

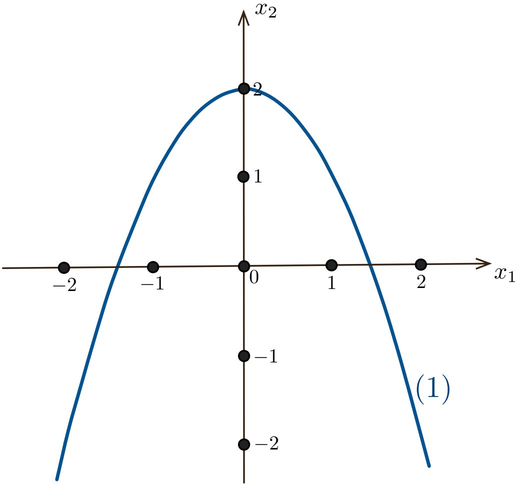
Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \\ & g_i(x) \leq 0 \qquad (i = 1, \ldots, k) \end{array}$$

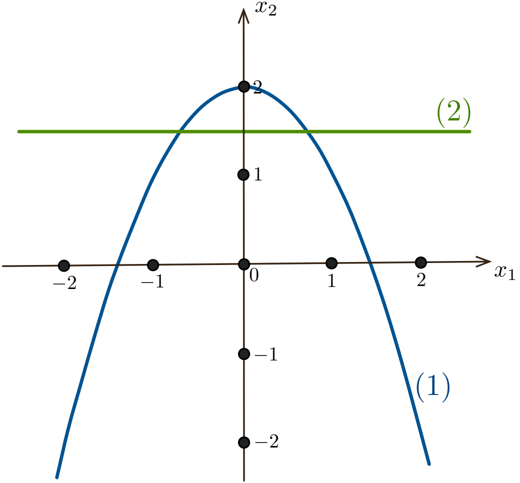
$$\begin{array}{lrcl} \min & x_2 & & \\ \text{s.t.} & & & \\ & -x_1^2 - x_2 + 2 & \leq & 0 \\ & x_2 - \frac{3}{2} & \leq & 0 \\ & x_1 - \frac{3}{2} & \leq & 0 \\ & \underbrace{-x_1 - 2}_{g_4(x)} & \leq & 0 \end{array}$$

$$\begin{array}{ll} \min & x_2 \\ \text{s.t.} & \\ & -x_1^2 - x_2 + 2 \leq 0 \\ & x_2 - 3/2 \leq 0 \\ & x_1 - 3/2 \leq 0 \\ & -x_1 - 2 \leq 0 \end{array}$$



(1) $x_2 \geq 2 - x_1^2$.

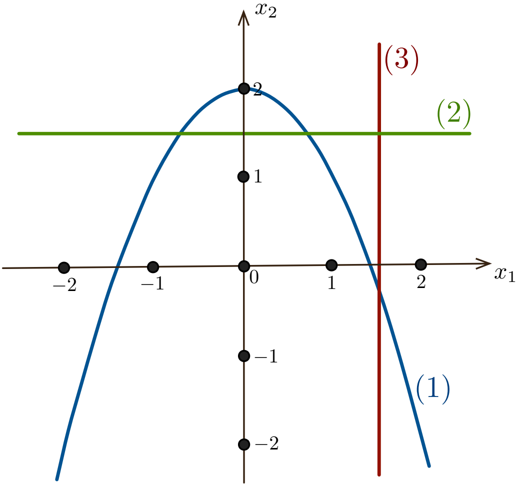
$$
\begin{array}{lrcl}
\min & x_2 & & \\
\text{s.t.} & & & \\
& -x_1^2 - x_2 + 2 & \leq & 0 \\
& x_2 - 3/2 & \leq & 0 \\
& x_1 - 3/2 & \leq & 0 \\
& -x_1 - 2 & \leq & 0
\end{array}
$$



(1) $x_2 \geq 2 - x_1^2$.

(2) $x_2 \leq \frac{3}{2}$.

$$\begin{array}{lrcl} \min & x_2 & & \\ \text{s.t.} & & & \\ & -x_1^2 - x_2 + 2 & \leq & 0 \\ & x_2 - 3/2 & \leq & 0 \\ & x_1 - 3/2 & \leq & 0 \\ & -x_1 - 2 & \leq & 0 \end{array}$$

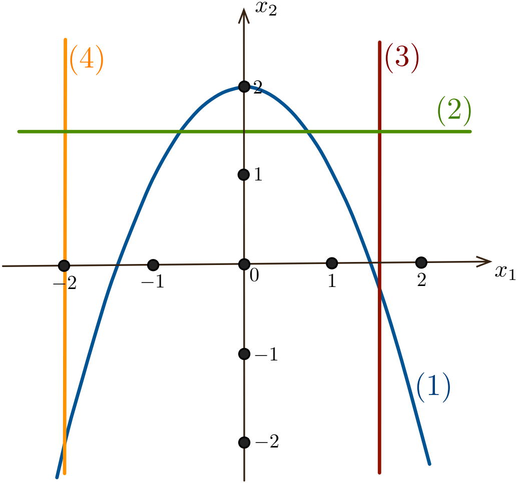


(1) $x_2 \geq 2 - x_1^2$.

(2) $x_2 \leq \frac{3}{2}$.

(3) $x_1 \leq \frac{3}{2}$.

$$
\begin{array}{lrcl}
\min & x_2 & & \\
\text{s.t.} & & & \\
& -x_1^2 - x_2 + 2 & \leq & 0 \\
& x_2 - 3/2 & \leq & 0 \\
& x_1 - 3/2 & \leq & 0 \\
& -x_1 - 2 & \leq & 0
\end{array}
$$



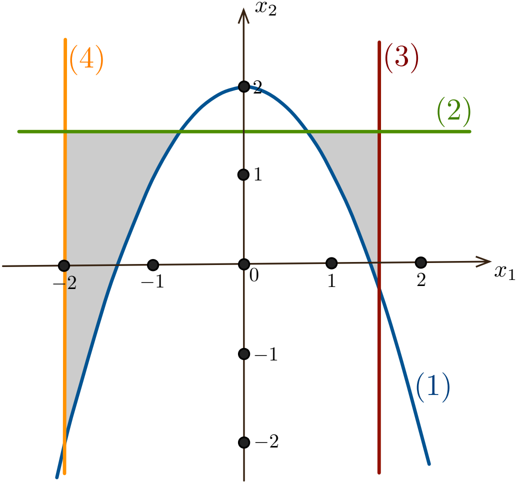
(1) $x_2 \geq 2 - x_1^2$.

(2) $x_2 \leq \frac{3}{2}$.

(3) $x_1 \leq \frac{3}{2}$.

(4) $x_1 \geq -2$.

$$
\begin{array}{lrcl}
\min & x_2 & & \\
\text{s.t.} & & & \\
 & -x_1^2 - x_2 + 2 & \leq & 0 \\
 & x_2 - 3/2 & \leq & 0 \\
 & x_1 - 3/2 & \leq & 0 \\
 & -x_1 - 2 & \leq & 0
\end{array}
$$



Feasible region

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\boxed{\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \qquad \text{(P)}$$

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\boxed{\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \tag{P}$$

Remark

We may assume $f(x)$ is a linear function, i.e., $f(x) = c^\top x$.

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\begin{array}{|lll|} \hline \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \\ \hline \end{array} \tag{P}$$

Remark

We may assume $f(x)$ is a linear function, i.e., $f(x) = c^\top x$.

We can rewrite (P) as

$$\begin{array}{|lll|} \hline \min & \lambda & \\ \text{s.t.} & & \\ & \lambda \geq f(x) & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \\ \hline \end{array} \tag{Q}$$

Definition

A Nonlinear Program (NLP) is a problem of the form:

$$\boxed{\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \qquad \text{(P)}$$

Remark

We may assume $f(x)$ is a linear function, i.e., $f(x) = c^\top x$.

We can rewrite (P) as

$$\boxed{\begin{array}{lll} \min & \lambda & \\ \text{s.t.} & & \\ & \lambda \geq f(x) & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \qquad \text{(Q)}$$

The optimal solution to (Q) will have $\lambda = f(x)$.

Nonlinear Programs Generalize Linear Programs

$$\max \quad x_1 + x_2$$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &= 4 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Nonlinear Programs Generalize Linear Programs

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & \\
& 2x_1 - x_2 \geq 3 \\
& x_1 - x_2 = 4 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & -x_1 - x_2 \\
\text{s.t.} \quad & \\
& -2x_1 + x_2 + 3 \leq 0 \\
& x_1 - x_2 - 4 \leq 0 \\
& -x_1 + x_2 + 4 \leq 0 \\
& -x_1 \leq 0 \\
& -x_2 \leq 0
\end{aligned}
$$

Nonlinear Programs Generalize Linear Programs

$$\max \quad x_1 + x_2$$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &= 4 \end{aligned}$$

$$x_1, x_2 \geq 0$$

$$\min \quad -x_1 - x_2$$

s.t.

$$\begin{aligned} -2x_1 + x_2 + 3 &\leq 0 \\ x_1 - x_2 - 4 &\leq 0 \\ -x_1 + x_2 + 4 &\leq 0 \\ -x_1 &\leq 0 \\ -x_2 &\leq 0 \end{aligned}$$

Nonlinear Programs can also generalize INTEGER PROGRAMS!

Nonlinear Programs Generalize Integer Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \in \{0,1\} \quad (j = 1, \ldots, n) \end{array}$$

$0, 1$ IP

Nonlinear Programs Generalize Integer Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \in \{0,1\} \quad (j = 1, \ldots, n) \end{array}$$

$0, 1$ IP

Idea

$x_j \in \{0,1\}$

Nonlinear Programs Generalize **Integer** Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \in \{0,1\} \quad (j = 1, \ldots, n) \end{array}$$

$0, 1$ IP

Idea

$x_j \in \{0,1\} \quad \Longleftrightarrow \quad x_j(1 - x_j) = 0$

Nonlinear Programs Generalize Integer Programs

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & \\ & Ax \leq b \\ & x_j \in \{0,1\} \quad (j = 1, \ldots, n) \end{aligned}$$

$0, 1$ IP

Idea

$x_j \in \{0,1\} \quad \Longleftrightarrow \quad x_j(1 - x_j) = 0$

$$\begin{aligned} \min \quad & -c^\top x \\ \text{s.t.} \quad & \\ & \begin{array}{rcll} Ax & \leq & b & \\ x_j(1-x_j) & \leq & 0 & (j = 1, \ldots, n) \\ -x_j(1-x_j) & \leq & 0 & (j = 1, \ldots, n) \end{array} \end{aligned}$$

Quadratic NLP

Nonlinear Programs Generalize **Integer** Programs

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & \\ & Ax \le b \\ & x_j \in \{0,1\} \quad (j = 1, \ldots, n) \end{aligned}$$

$0, 1$ IP

Idea

$x_j \in \{0,1\} \quad \Longleftrightarrow \quad x_j(1 - x_j) = 0$

$$\begin{aligned} \min \quad & -c^\top x \\ \text{s.t.} \quad & \\ & \begin{array}{rcll} Ax & \le & b & \\ x_j(1-x_j) & \le & 0 & (j = 1, \ldots, n) \\ -x_j(1-x_j) & \le & 0 & (j = 1, \ldots, n) \end{array} \end{aligned}$$

Quadratic NLP

Remark

$0, 1$ IPs are hard to solve; thus, quadratic NLPs are also hard to solve.

Nonlinear Programs Generalize **Integer** Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \text{ integer} \quad (j = 1, \ldots, n) \end{array}$$

pure IP

Nonlinear Programs Generalize **Integer** Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \text{ integer} \quad (j = 1, \ldots, n) \end{array}$$

pure IP

Idea

x_j integer

Nonlinear Programs Generalize Integer Programs

$$
\begin{array}{ll}
\max & c^\top x \\
\text{s.t.} & \\
& Ax \le b \\
& x_j \text{ integer} \quad (j = 1, \ldots, n)
\end{array}
$$

pure IP

Idea

x_j integer $\iff$ $\sin(\pi x_j) = 0.$

Nonlinear Programs Generalize Integer Programs

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax \le b \\ & x_j \text{ integer} \quad (j = 1, \ldots, n) \end{array}$$

pure IP

Idea

x_j integer $\iff$ $\sin(\pi x_j) = 0$.

$$\begin{array}{lrcll} \min & -c^\top x & & & \\ \text{s.t.} & & & & \\ & Ax & \le & b & \\ & \sin(\pi x_j) & = & 0 & (j = 1, \ldots, n) \end{array}$$

Nonlinear Programs Generalize Integer Programs

$$
\begin{array}{ll}
\max & c^\top x \\
\text{s.t.} & \\
& Ax \leq b \\
& x_j \text{ integer} \quad (j = 1, \ldots, n)
\end{array}
$$

pure IP

Idea

x_j integer $\iff$ $\sin(\pi x_j) = 0$.

$$
\begin{array}{lrcll}
\min & -c^\top x & & & \\
\text{s.t.} & & & & \\
& Ax & \leq & b & \\
& \sin(\pi x_j) & = & 0 & (j = 1, \ldots, n)
\end{array}
$$

Remark

IPs are hard to solve; thus, NLPs are also hard to solve.

Question

What makes solving an NLP hard?

Question

What makes solving an NLP hard?

META STRATEGY FOR SOLVING AN OPTIMIZATION PROBLEM

Question

What makes solving an NLP hard?

Meta strategy for solving an optimization problem

- Find a feasible solution x.

Question

What makes solving an NLP hard?

META STRATEGY FOR SOLVING AN OPTIMIZATION PROBLEM

- Find a feasible solution x.
- If x is optimal, STOP.

Question

What makes solving an NLP hard?

META STRATEGY FOR SOLVING AN OPTIMIZATION PROBLEM

- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.

Question

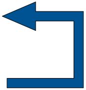
What makes solving an NLP hard?

Meta strategy for solving an optimization problem

- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.

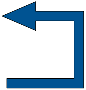
Meta strategy for solving an optimization problem

- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.



Meta strategy for solving an optimization problem

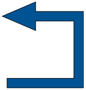
- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.



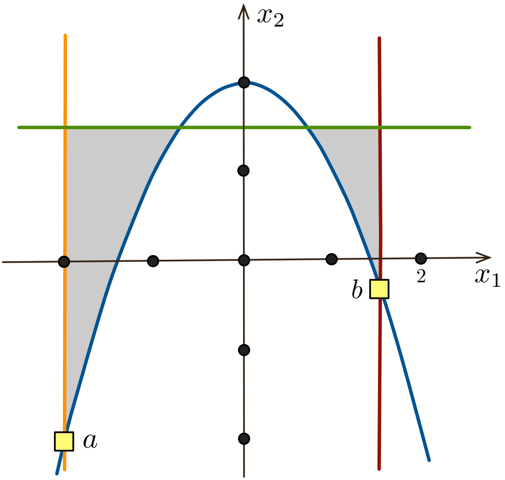
$$
\begin{aligned}
\min \quad & x_2 \\
\text{s.t.} \quad & \\
-x_1^2 - x_2 + 2 &\leq 0 \\
x_2 - 3/2 &\leq 0 \\
x_1 - 3/2 &\leq 0 \\
-x_1 - 2 &\leq 0
\end{aligned}
$$

Meta strategy for solving an optimization problem

- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.

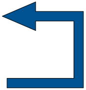


$$\begin{aligned}
\min \quad & x_2 \\
\text{s.t.} \quad & \\
-x_1^2 - x_2 + 2 &\leq 0 \\
x_2 - 3/2 &\leq 0 \\
x_1 - 3/2 &\leq 0 \\
-x_1 - 2 &\leq 0
\end{aligned}$$

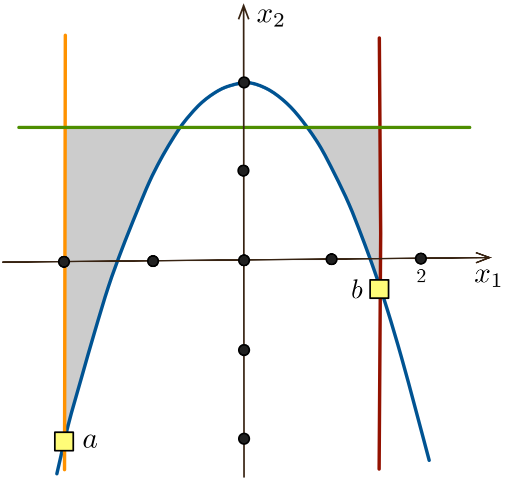


Meta strategy for solving an optimization problem

- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.

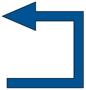


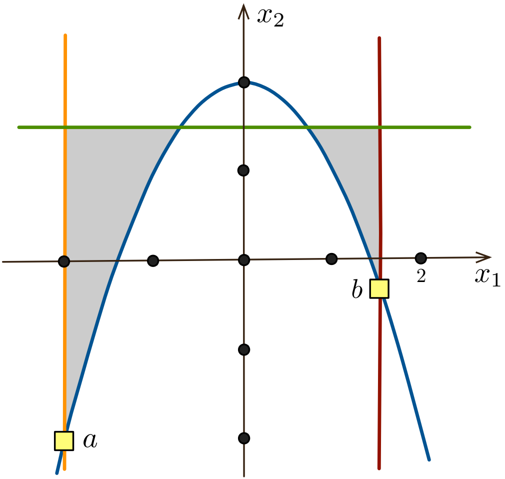
a is an optimal solution



Meta strategy for solving an optimization problem

- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.





a is an optimal solution

no better solution around b

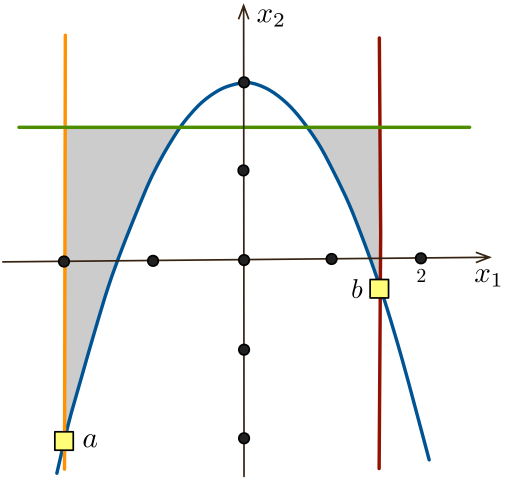
Meta strategy for solving an optimization problem

- Find a feasible solution x.
- If x is optimal, STOP.
- Starting with x, find a "better" feasible solution.

a is an optimal solution

no better solution around b

b is a local optimum



Definition

Consider

$$\min \{f(x) : x \in S\}. \tag{P}$$

Definition

Consider

$$\min \{f(x) : x \in S\}. \tag{P}$$

$x \in S$ is a local optimum

Definition

Consider

$$\min \{f(x) : x \in S\}. \tag{P}$$

$x \in S$ is a local optimum if there exists $\delta > 0$ such that

Definition

Consider

$$\min \{ f(x) : x \in S \} . \tag{P}$$

$x \in S$ is a local optimum if there exists $\delta > 0$ such that

$$\forall x' \in S \quad \text{where} \quad ||x' - x|| \leq \delta \quad \text{we have} \quad f(x) \leq f(x').$$

Definition

Consider

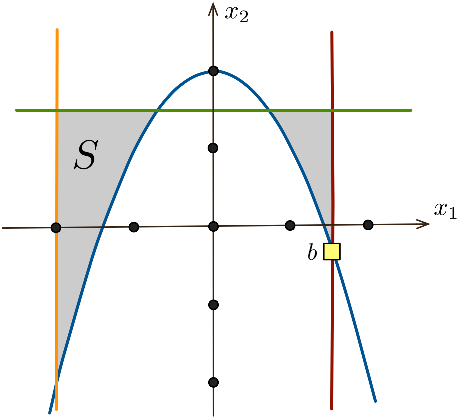
$$\min\{f(x) : x \in S\}. \tag{P}$$

$x \in S$ is a local optimum if there exists $\delta > 0$ such that

$$\forall x' \in S \quad \text{where} \quad ||x' - x|| \leq \delta \quad \text{we have} \quad f(x) \leq f(x').$$

$\min\{x_2 : x \in S\}$

b is a local optimum



Definition

Consider

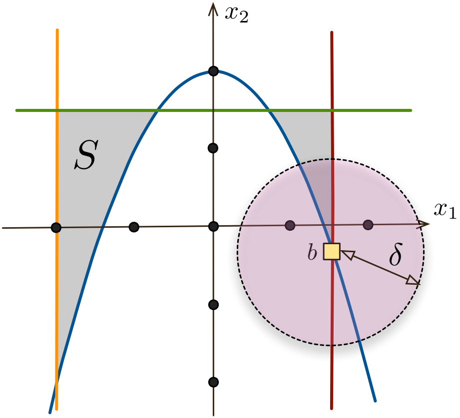
$$\min \{f(x) : x \in S\}. \qquad \text{(P)}$$

$x \in S$ is a local optimum if there exists $\delta > 0$ such that

$$\forall x' \in S \quad \text{where} \quad ||x' - x|| \leq \delta \quad \text{we have} \quad f(x) \leq f(x').$$

$\min\{x_2 : x \in S\}$

b is a local optimum



Convexity Helps

Definition

Consider

$$\min \{f(x) : x \in S\}. \qquad \text{(P)}$$

$x \in S$ is a local optimum if there exists $\delta > 0$ such that

$$\forall x' \in S \quad \text{where} \quad ||x' - x|| \leq \delta \quad \text{we have} \quad f(x) \leq f(x').$$

Convexity Helps

Definition

Consider

$$\min\{f(x) : x \in S\}. \qquad \text{(P)}$$

$x \in S$ is a local optimum if there exists $\delta > 0$ such that

$$\forall x' \in S \quad \text{where} \quad ||x' - x|| \leq \delta \quad \text{we have} \quad f(x) \leq f(x').$$

Proposition

Consider

$$\min\{c^\top x : x \in S\}. \qquad \text{(P)}$$

Convexity Helps

Definition

Consider

$$\min\left\{f(x) : x \in S\right\}. \tag{P}$$

$x \in S$ is a local optimum if there exists $\delta > 0$ such that

$$\forall x' \in S \quad \text{where} \quad ||x' - x|| \leq \delta \quad \text{we have} \quad f(x) \leq f(x').$$

Proposition

Consider

$$\min\left\{c^\top x : x \in S\right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

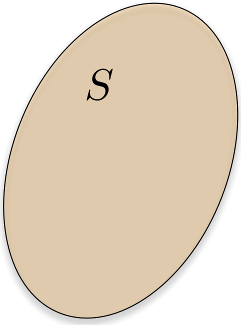
Proposition

Consider

$$\min\left\{c^\top x : x \in S\right\}. \qquad \text{(P)}$$

If S is convex and x is a local optimum, then x is optimal.

Proof



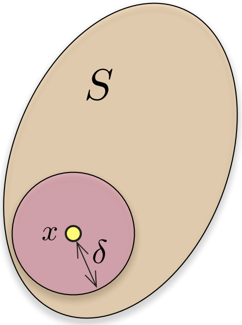
Proposition

Consider

$$\min\left\{c^\top x : x \in S\right\}. \qquad \text{(P)}$$

If S is convex and x is a local optimum, then x is optimal.

Proof



Proposition

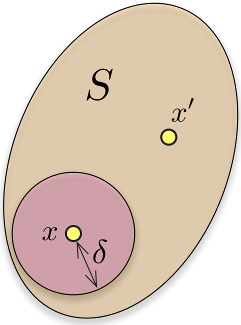
Consider

$$\min \left\{ c^\top x : x \in S \right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.



Proposition

Consider

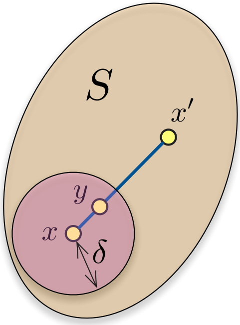
$$\min\left\{c^\top x : x \in S\right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

Let $y = \lambda x' + (1-\lambda)x$ for $\lambda > 0$ small.



Proposition

Consider

$$\min\left\{c^{\top}x : x \in S\right\}. \quad \text{(P)}$$

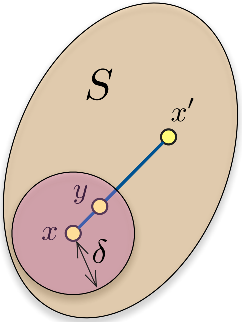
If S is convex and x is a local optimum, then x is optimal.

Proof

Suppose $\exists x' \in S$ with $c^{\top}x' < c^{\top}x$.

Let $y = \lambda x' + (1-\lambda)x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.



Proposition

Consider

$$\min \left\{ c^\top x : x \in S \right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

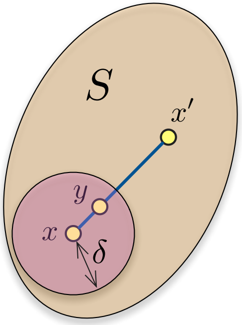
Proof

Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

Let $y = \lambda x' + (1 - \lambda) x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.

Since S is convex, $y \in S$.



Proposition

Consider

$$\min\left\{c^\top x : x \in S\right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

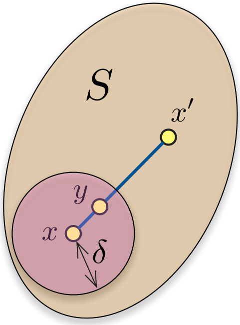
Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

Let $y = \lambda x' + (1-\lambda)x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.

Since S is convex, $y \in S$.

$$c^\top y$$



Proposition

Consider

$$\min \left\{ c^\top x : x \in S \right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

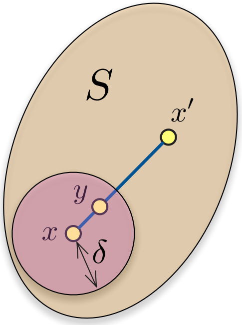
Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

Let $y = \lambda x' + (1 - \lambda)x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.

Since S is convex, $y \in S$.

$$c^\top y = c^\top (\lambda x' + (1 - \lambda)x)$$



Proposition

Consider

$$\min\left\{c^\top x : x \in S\right\}. \qquad \text{(P)}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

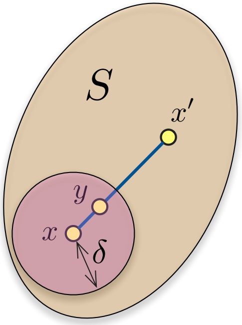
Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

Let $y = \lambda x' + (1-\lambda)x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.

Since S is convex, $y \in S$.

$$\begin{aligned} c^\top y &= c^\top(\lambda x' + (1-\lambda)x) \\ &= \underbrace{\lambda}_{>0}\underbrace{c^\top x'}_{<c^\top x} + (1-\lambda)c^\top x \end{aligned}$$



Proposition

Consider

$$\min\left\{c^\top x : x \in S\right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

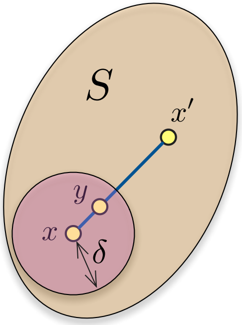
Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

Let $y = \lambda x' + (1-\lambda)x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.

Since S is convex, $y \in S$.

$$\begin{aligned} c^\top y &= c^\top(\lambda x' + (1-\lambda)x) \\ &= \underbrace{\lambda}_{>0}\underbrace{c^\top x'}_{<c^\top x} + (1-\lambda)c^\top x \\ &< \lambda c^\top x + (1-\lambda)c^\top x \end{aligned}$$



Proposition

Consider

$$\min\left\{c^\top x : x \in S\right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

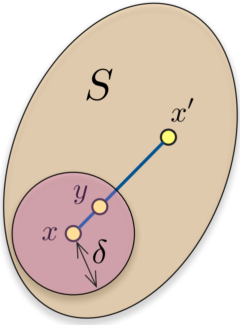
Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

Let $y = \lambda x' + (1-\lambda)x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.

Since S is convex, $y \in S$.

$$\begin{aligned} c^\top y &= c^\top(\lambda x' + (1-\lambda)x) \\ &= \underbrace{\lambda}_{>0}\underbrace{c^\top x'}_{<c^\top x} + (1-\lambda)c^\top x \\ &< \lambda c^\top x + (1-\lambda)c^\top x \\ &= c^\top x \end{aligned}$$



Proposition

Consider

$$\min \left\{ c^\top x : x \in S \right\}. \tag{P}$$

If S is convex and x is a local optimum, then x is optimal.

Proof

Suppose $\exists x' \in S$ with $c^\top x' < c^\top x$.

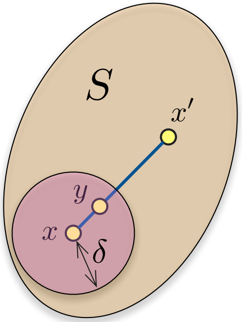
Let $y = \lambda x' + (1-\lambda)x$ for $\lambda > 0$ small.

As λ small $||y - x|| \leq \delta$.

Since S is convex, $y \in S$.

$$\begin{aligned} c^\top y &= c^\top(\lambda x' + (1-\lambda)x) \\ &= \underbrace{\lambda}_{>0} \underbrace{c^\top x'}_{<c^\top x} + (1-\lambda)c^\top x \\ &< \lambda c^\top x + (1-\lambda)c^\top x \\ &= c^\top x \end{aligned}$$

A contradiction.



Consider

$$\boxed{\begin{array}{lll} \min & c^\top x & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \tag{P}$$

Consider

$$\boxed{\begin{array}{lll} \min & c^\top x & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \qquad \text{(P)}$$

Goal: Study a case where the feasible region of (P) is convex.

Consider

$$\boxed{\begin{array}{lll} \min & c^\top x & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \tag{P}$$

Goal: Study a case where the feasible region of (P) is convex.

- We will define convex functions

Consider

$$\boxed{\begin{array}{lll} \min & c^\top x & \\ \text{s.t.} & & \\ & g_i(x) \le 0 & (i = 1, \dots, k) \end{array}} \qquad \text{(P)}$$

Goal: Study a case where the feasible region of (P) is convex.

- We will define convex functions
- We will prove

Proposition

If $g_1, \dots, g_k$ are all convex, then the feasible region of (P) is convex.

Definition

Function $f : \Re^n \rightarrow \Re$ is convex

Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda) f(b)$$

Definition

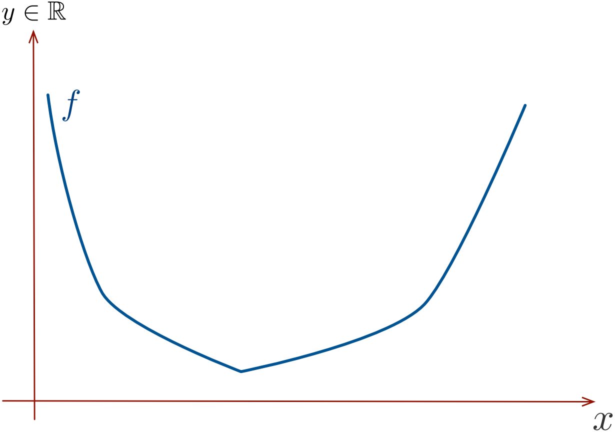
Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda) b\big) \le \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \le \lambda \le 1.$$

Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda) b\big) \leq \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$

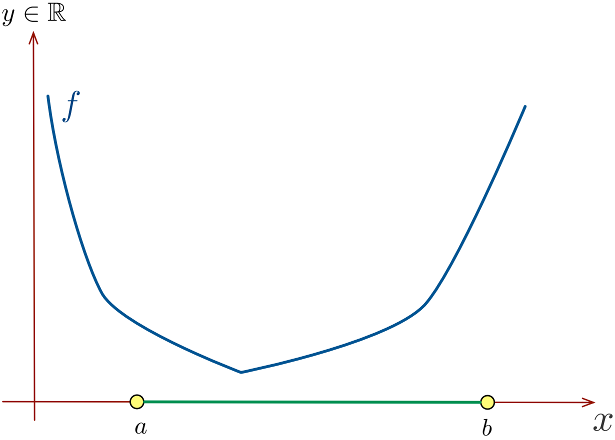


Convex function!

Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda) b\big) \leq \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$

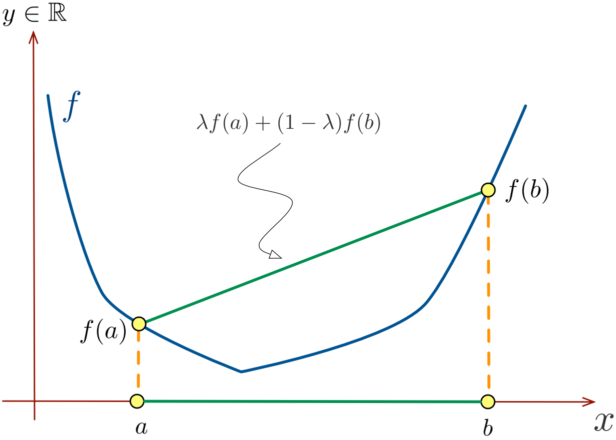


Convex function!

Definition

Function $f : \Re^n \rightarrow \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$



Convex function!

Definition

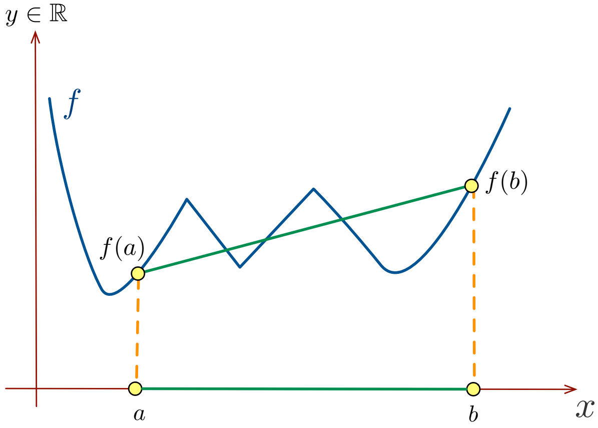
Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda)b\big) \le \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \le \lambda \le 1.$$

Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

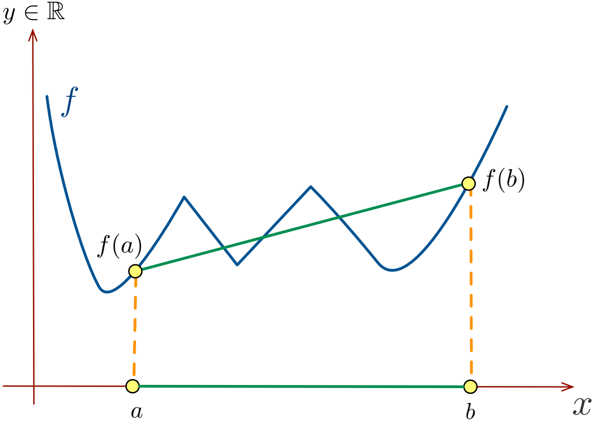
$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$



Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$



Not a convex function!

Definition

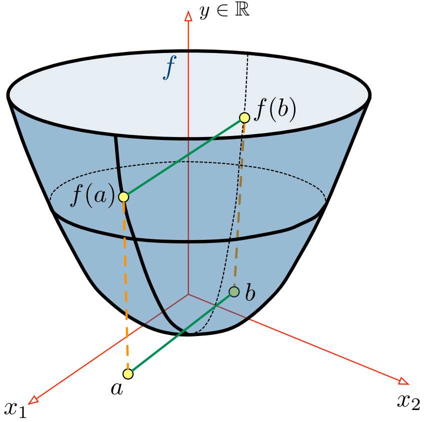
Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$

Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

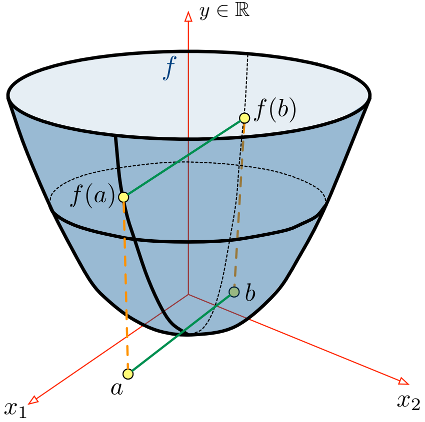
$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda)f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$



Definition

Function $f : \Re^n \to \Re$ is convex if for all $a, b \in \Re^n$,

$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda) f(b) \quad \text{for all} \quad 0 \leq \lambda \leq 1.$$



Convex function!

Example

We claim that $f(x) = x^2$ is convex.

Example

We claim that $f(x) = x^2$ is convex.

Pick $a, b \in \Re$ and pick λ where $0 \leq \lambda \leq 1$.

Example

We claim that $f(x) = x^2$ is convex.

Pick $a, b \in \Re$ and pick λ where $0 \leq \lambda \leq 1$.

To check:

$$[\lambda a + (1-\lambda)b]^2 \stackrel{?}{\leq} \lambda a^2 + (1-\lambda)b^2.$$

Example

We claim that $f(x) = x^2$ is convex.

Pick $a, b \in \Re$ and pick λ where $0 \leq \lambda \leq 1$.

To check:

$$[\lambda a + (1-\lambda) b]^2 \overset{?}{\leq} \lambda a^2 + (1-\lambda) b^2.$$

We may assume that $\lambda \neq 0, 1$.

Example

We claim that $f(x) = x^2$ is convex.

Pick $a, b \in \Re$ and pick λ where $0 \leq \lambda \leq 1$.

To check:

$$[\lambda a + (1-\lambda)b]^2 \stackrel{?}{\leq} \lambda a^2 + (1-\lambda)b^2.$$

We may assume that $\lambda \neq 0, 1$.

After simplifying

$$\lambda(1-\lambda)2ab \stackrel{?}{\leq} \lambda(1-\lambda)(a^2 + b^2),$$

Example

We claim that $f(x) = x^2$ is convex.

Pick $a, b \in \Re$ and pick λ where $0 \leq \lambda \leq 1$.

<u>To check:</u>

$$[\lambda a + (1-\lambda)b]^2 \overset{?}{\leq} \lambda a^2 + (1-\lambda)b^2.$$

We may assume that $\lambda \neq 0, 1$.

After simplifying

$$\lambda(1-\lambda)2ab \overset{?}{\leq} \lambda(1-\lambda)(a^2+b^2),$$

or, equivalently, as $\lambda, (1-\lambda) > 0$,

$$a^2 + b^2 - 2ab \overset{?}{\geq} 0,$$

Example

We claim that $f(x) = x^2$ is convex.

Pick $a, b \in \Re$ and pick λ where $0 \leq \lambda \leq 1$.

To check:

$$[\lambda a + (1-\lambda)b]^2 \overset{?}{\leq} \lambda a^2 + (1-\lambda)b^2.$$

We may assume that $\lambda \neq 0, 1$.

After simplifying

$$\lambda(1-\lambda)2ab \overset{?}{\leq} \lambda(1-\lambda)(a^2 + b^2),$$

or, equivalently, as $\lambda, (1-\lambda) > 0$,

$$a^2 + b^2 - 2ab \overset{?}{\geq} 0,$$

which is the case as $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$.

Why Do We Care About Convex Functions?

Proposition

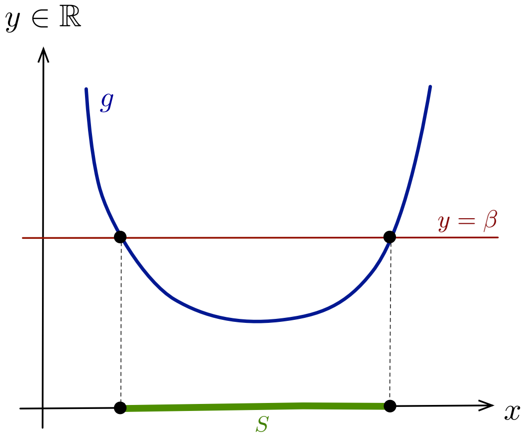
Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

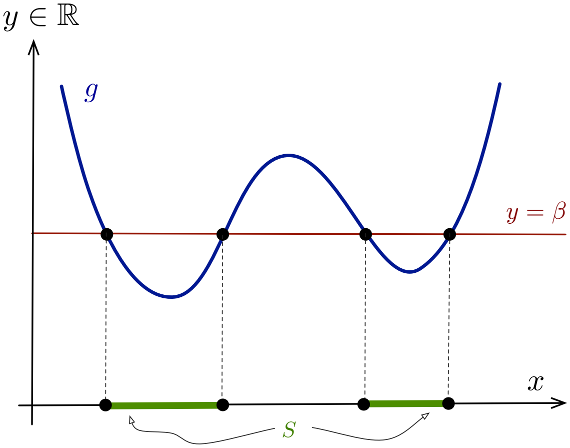
Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.



Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

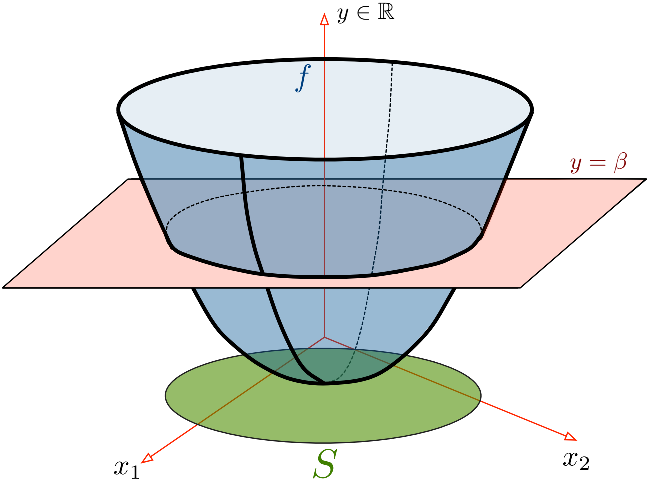
Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.



Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.



Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \le \beta\}$ is a convex set.

Proof

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \le \beta\}$ is a convex set.

Proof

Pick $a, b \in S$.

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.

Proof

Pick $a, b \in S$.

Pick λ where $0 \leq \lambda \leq 1$.

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.

Proof

Pick $a, b \in S$.

Pick λ where $0 \leq \lambda \leq 1$.

Let $x = \lambda a + (1 - \lambda)b$.

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.

Proof

Pick $a, b \in S$.

Pick λ where $0 \leq \lambda \leq 1$.

Let $x = \lambda a + (1 - \lambda)b$.

Our goal is to show that $x \in S$, i.e., that $g(x) \leq \beta$.

Proposition

Let $g : \Re^n \to \Re$ be a convex <u>function</u> and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex <u>set</u>.

Proof

Pick $a, b \in S$.

Pick λ where $0 \leq \lambda \leq 1$.

Let $x = \lambda a + (1 - \lambda)b$.

Our goal is to show that $x \in S$, i.e., that $g(x) \leq \beta$.

$$g(x) \quad = \quad g\left(\lambda a + (1 - \lambda)b\right)$$

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \le \beta\}$ is a convex set.

Proof

Pick $a, b \in S$.

Pick λ where $0 \le \lambda \le 1$.

Let $x = \lambda a + (1 - \lambda)b$.

Our goal is to show that $x \in S$, i.e., that $g(x) \le \beta$.

$$\begin{aligned} g(x) &= g\left(\lambda a + (1-\lambda)b\right) \\ &\le \lambda g(a) + (1-\lambda)g(b) \qquad \text{(convexity of } g\text{)} \end{aligned}$$

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.

Proof

Pick $a, b \in S$.

Pick λ where $0 \leq \lambda \leq 1$.

Let $x = \lambda a + (1 - \lambda) b$.

Our goal is to show that $x \in S$, i.e., that $g(x) \leq \beta$.

$$
\begin{aligned}
g(x) &= g\left(\lambda a + (1 - \lambda) b\right) \\
&\leq \underbrace{\lambda}_{\geq 0} \underbrace{g(a)}_{\leq \beta} + \underbrace{(1 - \lambda)}_{\geq 0} \underbrace{g(b)}_{\leq \beta} \qquad \text{(since } a, b \in S\text{)}
\end{aligned}
$$

Proposition

Let $g : \Re^n \to \Re$ be a convex function and $\beta \in \Re$.

Then $S = \{x \in \Re^n : g(x) \leq \beta\}$ is a convex set.

Proof

Pick $a, b \in S$.

Pick λ where $0 \leq \lambda \leq 1$.

Let $x = \lambda a + (1-\lambda)b$.

Our goal is to show that $x \in S$, i.e., that $g(x) \leq \beta$.

$$\begin{aligned} g(x) &= g\left(\lambda a + (1-\lambda)b\right) \\ &\leq \underbrace{\lambda}_{\geq 0} \underbrace{g(a)}_{\leq \beta} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{g(b)}_{\leq \beta} \qquad (\text{since } a, b \in S) \\ &\leq \lambda\beta + (1-\lambda)\beta = \beta. \end{aligned}$$

Proposition

$$\boxed{\begin{array}{lll} \min & c^\top x & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \qquad \text{(P)}$$

Proposition

$$\boxed{\begin{array}{lll} \min & c^\top x & \\ \text{s.t.} & & \\ & g_i(x) \leq 0 & (i = 1, \ldots, k) \end{array}} \tag{P}$$

If all functions g_i are convex, then the feasible region of (P) is convex.

Proposition

$$\boxed{\begin{aligned} &\min \quad c^\top x \\ &\text{s.t.} \\ &\qquad\quad g_i(x) \leq 0 \qquad (i = 1, \dots, k) \end{aligned}} \tag{P}$$

If all functions g_i are convex, then the feasible region of (P) is convex.

Proof

Proposition

$$\boxed{\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \\ & g_i(x) \le 0 \qquad (i = 1, \ldots, k) \end{array}} \qquad \text{(P)}$$

If all functions g_i are convex, then the feasible region of (P) is convex.

Proof

Let $S_i = \{x : g_i(x) \le 0\}$.

Proposition

$$\boxed{\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \\ & g_i(x) \le 0 \qquad (i = 1, \ldots, k) \end{array}} \tag{P}$$

If all functions g_i are convex, then the feasible region of (P) is convex.

Proof

Let $S_i = \{x : g_i(x) \le 0\}$.

By the previous result, S_i is convex.

Proposition

$$\boxed{\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \\ & g_i(x) \leq 0 \qquad (i = 1, \ldots, k) \end{array}} \tag{P}$$

If all functions g_i are convex, then the feasible region of (P) is convex.

Proof

Let $S_i = \{x : g_i(x) \leq 0\}$.

By the previous result, S_i is convex.

The feasible region of (P) is $S_1 \cap S_2 \cap \ldots \cap S_k$.

Proposition

$$\boxed{\begin{array}{lll} \min & c^\top x & \\ \text{s.t.} & & \\ & g_i(x) \le 0 & (i = 1, \ldots, k) \end{array}} \qquad \text{(P)}$$

If all functions g_i are convex, then the feasible region of (P) is convex.

Proof

Let $S_i = \{x : g_i(x) \le 0\}$.

By the previous result, S_i is convex.

The feasible region of (P) is $S_1 \cap S_2 \cap \ldots \cap S_k$.

Since the intersection of convex sets is convex, the result follows.

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function.

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

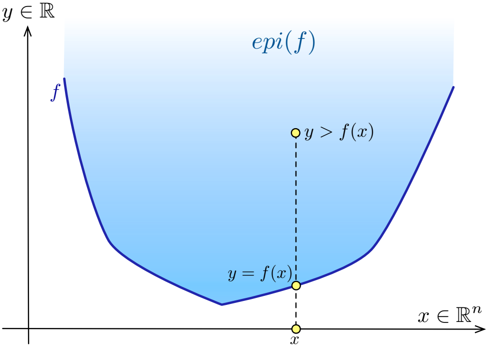
$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$

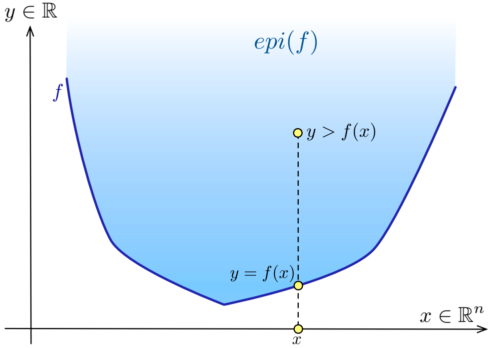


Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$



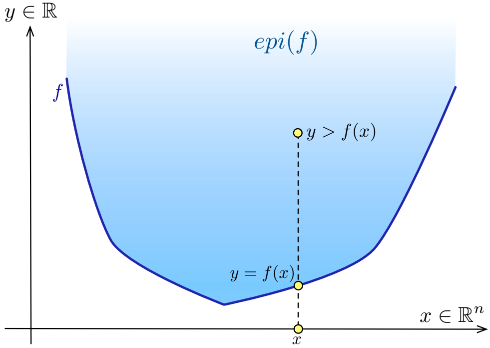
f is convex

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$



f is convex

$epi(f)$ is convex

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

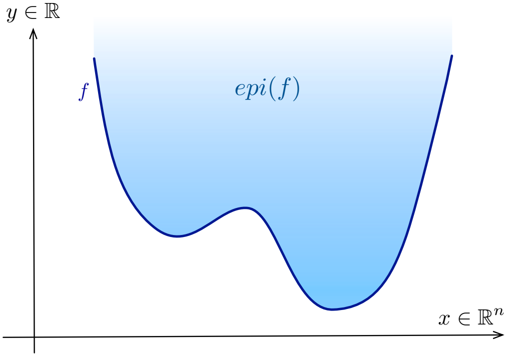
$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$

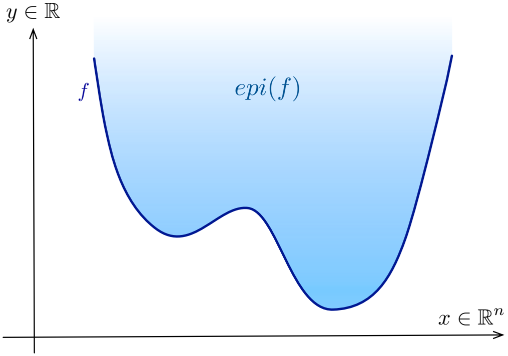


Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$



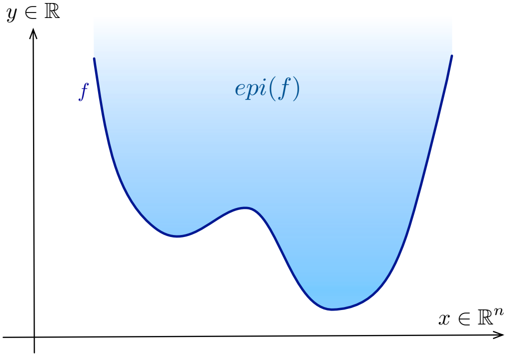
f is NOT convex

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$



f is NOT convex

$epi(f)$ is NOT convex

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

Convex Functions Versus Convex Sets

Definition

Let $f : \Re^n \to \Re$ be a function. The epigraph of f is then given by,

$$epi(f) = \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : x \in \Re^n, y \in \Re, y \geq f(x) \right\} \subseteq \Re^{n+1}.$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. f is convex $\implies$ $epi(f)$ is convex.
2. $epi(f)$ is convex $\implies$ f is convex.

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad$ f is convex.

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad$ f is convex.

Proof

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. f is convex $\implies$ $epi(f)$ is convex.
2. $epi(f)$ is convex $\implies$ f is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$.

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. f is convex $\implies$ $epi(f)$ is convex.
2. $epi(f)$ is convex $\implies$ f is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \leq \lambda \leq 1$.

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad f$ is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \leq \lambda \leq 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix} \alpha \\ a \end{pmatrix} + (1 - \lambda) \begin{pmatrix} \beta \\ b \end{pmatrix}$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. f is convex $\Longrightarrow$ $epi(f)$ is convex.
2. $epi(f)$ is convex $\Longrightarrow$ f is convex.

Proof

Pick $\begin{pmatrix}\alpha \\ a\end{pmatrix} \begin{pmatrix}\beta \\ b\end{pmatrix} \in epi(f)$. Pick λ where $0 \le \lambda \le 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix}\alpha \\ a\end{pmatrix} + (1-\lambda)\begin{pmatrix}\beta \\ b\end{pmatrix} = \begin{pmatrix}\lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b\end{pmatrix} \qquad (\star)$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. f is convex $\implies$ $epi(f)$ is convex.
2. $epi(f)$ is convex $\implies$ f is convex.

Proof

Pick $\begin{pmatrix}\alpha \\ a\end{pmatrix} \begin{pmatrix}\beta \\ b\end{pmatrix} \in epi(f)$. Pick λ where $0 \leq \lambda \leq 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix}\alpha \\ a\end{pmatrix} + (1-\lambda)\begin{pmatrix}\beta \\ b\end{pmatrix} = \begin{pmatrix}\lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b\end{pmatrix} \qquad (\star)$$

Consider

$$f\left(\lambda a + (1-\lambda)b\right)$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad f$ is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \le \lambda \le 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix} \alpha \\ a \end{pmatrix} + (1-\lambda) \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} \lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b \end{pmatrix} \qquad (\star)$$

Consider

$$f(\lambda a + (1-\lambda)b) \quad \le \quad \text{(convexity of } f\text{)}$$
$$\lambda f(a) + (1-\lambda) f(b)$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad f$ is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \leq \lambda \leq 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix} \alpha \\ a \end{pmatrix} + (1-\lambda) \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} \lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b \end{pmatrix} \qquad (\star)$$

Consider

$$f\left(\lambda a + (1-\lambda)b\right) \quad \leq \quad \text{(convexity of } f\text{)}$$

$$\underbrace{\lambda}_{\geq 0} f(a) + \underbrace{(1-\lambda)}_{\geq 0} f(b)$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad f$ is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \le \lambda \le 1$.

<u>To show:</u> $epi(f)$ contains

$$\lambda \begin{pmatrix} \alpha \\ a \end{pmatrix} + (1-\lambda) \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} \lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b \end{pmatrix} \qquad (\star)$$

Consider

$$f\left(\lambda a + (1-\lambda) b\right) \quad \le \quad \text{(convexity of } f\text{)}$$

$$\underbrace{\lambda}_{\ge 0} \underbrace{f(a)}_{\le \alpha} + \underbrace{(1-\lambda)}_{\ge 0} \underbrace{f(b)}_{\le \beta}$$

Proposition

Let $f : \Re^n \to \Re$ be a function. Then

1. $\boxed{f \text{ is convex} \quad \Longrightarrow \quad epi(f) \text{ is convex.}}$
2. $epi(f)$ is convex $\quad \Longrightarrow \quad f$ is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \le \lambda \le 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix} \alpha \\ a \end{pmatrix} + (1-\lambda) \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} \lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b \end{pmatrix} \qquad (\star)$$

Consider

$$\begin{aligned} f\left(\lambda a + (1-\lambda)b\right) \quad & \le \quad \text{(convexity of } f\text{)} \\ \underbrace{\lambda}_{\ge 0} \underbrace{f(a)}_{\le \alpha} + \underbrace{(1-\lambda)}_{\ge 0} \underbrace{f(b)}_{\le \beta} \quad & \le \\ \lambda\alpha + (1-\lambda)\beta. & \end{aligned}$$

Proposition

Let $f : \Re^n \rightarrow \Re$ be a function. Then

1. f is convex $\Longrightarrow$ $epi(f)$ is convex.
2. $epi(f)$ is convex $\Longrightarrow$ f is convex.

Proof

Pick $\begin{pmatrix} \alpha \\ a \end{pmatrix} \begin{pmatrix} \beta \\ b \end{pmatrix} \in epi(f)$. Pick λ where $0 \leq \lambda \leq 1$.

To show: $epi(f)$ contains

$$\lambda \begin{pmatrix} \alpha \\ a \end{pmatrix} + (1-\lambda) \begin{pmatrix} \beta \\ b \end{pmatrix} = \begin{pmatrix} \lambda\alpha + (1-\lambda)\beta \\ \lambda a + (1-\lambda) b \end{pmatrix} \qquad (\star)$$

Consider

$$f\left(\lambda a + (1-\lambda) b\right) \leq \qquad \text{(convexity of } f\text{)}$$

$$\underbrace{\lambda}_{\geq 0} \underbrace{f(a)}_{\leq \alpha} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{f(b)}_{\leq \beta} \leq$$

$$\lambda\alpha + (1-\lambda)\beta.$$

Thus $(\star)$ is in $epi(f)$.

Recap

Recap

1. NLPs are hard in general.

Recap

1. NLPs are hard in general.
2. We may assume the objective function of NLPs is linear.

Recap

1. NLPs are hard in general.
2. We may assume the objective function of NLPs is linear.
3. Local optimum = optimal sol when the feasible region is convex.

Recap

1. NLPs are hard in general.
2. We may assume the objective function of NLPs is linear.
3. Local optimum = optimal sol when the feasible region is convex.
4. We defined convex functions.

Recap

1. NLPs are hard in general.
2. We may assume the objective function of NLPs is linear.
3. Local optimum = optimal sol when the feasible region is convex.
4. We defined convex functions.
5. Convex functions yield a convex feasible region.

Recap

1. NLPs are hard in general.
2. We may assume the objective function of NLPs is linear.
3. Local optimum $=$ optimal sol when the feasible region is convex.
4. We defined convex functions.
5. Convex functions yield a convex feasible region.
6. Convex functions and convex sets are related by epigraphs.