

Module 6: Nonlinear Programs (the KKT theorem)

$$\min \quad -x_1 - x_2$$

s.t.

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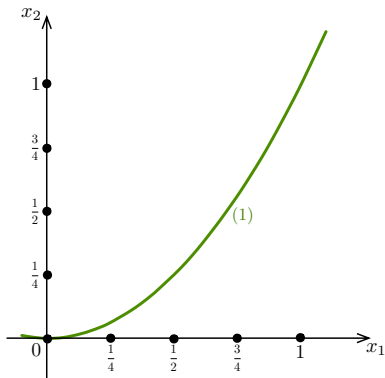
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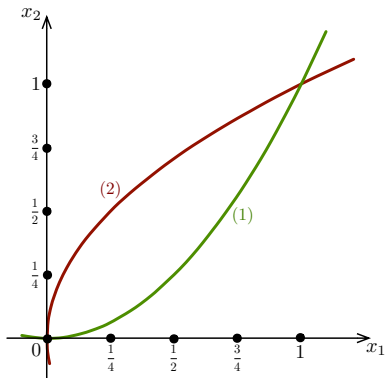
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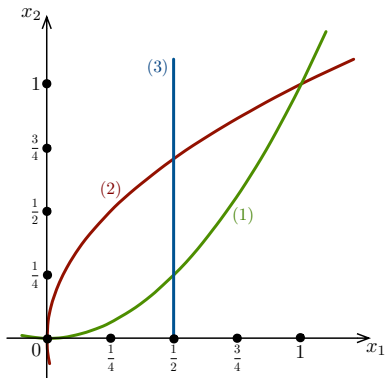
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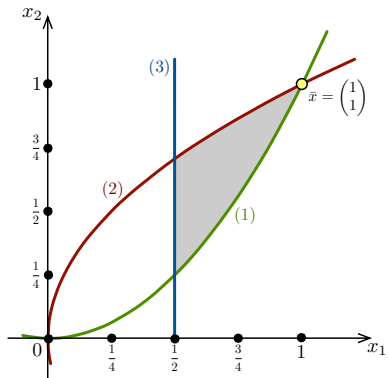
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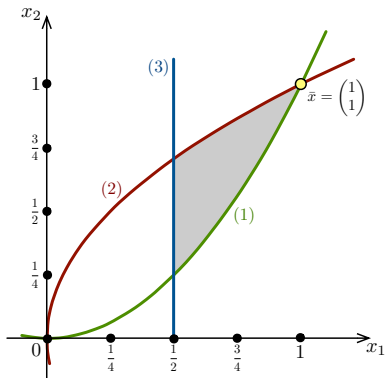
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Claim

$\bar{x} = (1, 1)^\top$ is an optimal solution to the NLP.

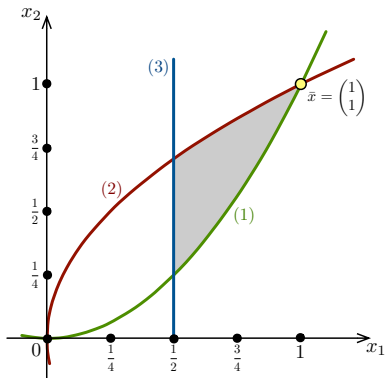
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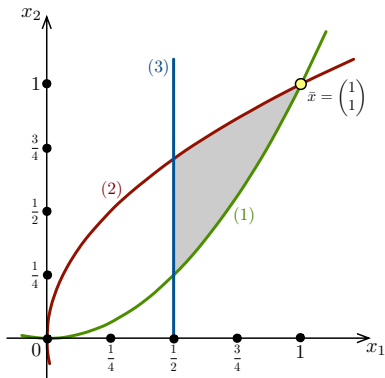
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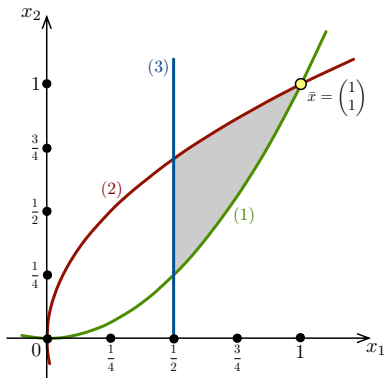
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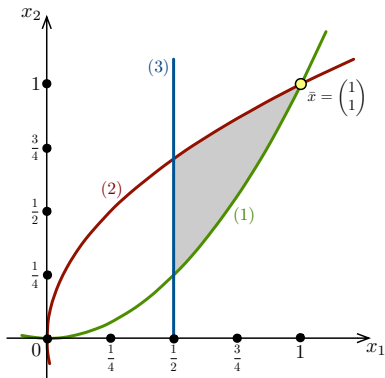
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Step 2. Prove \bar{x} is optimal for the relaxation.

Step 3. Deduce that \bar{x} is optimal for the NLP.

Original NLP

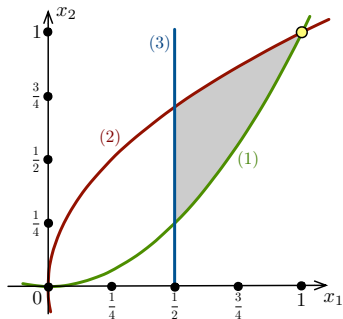
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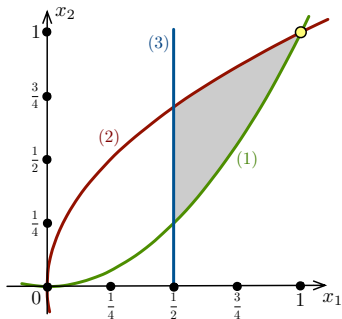
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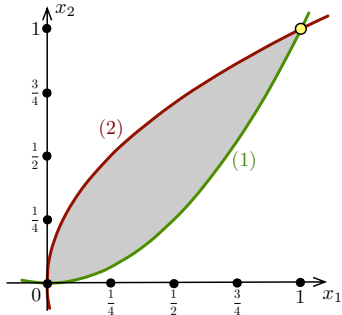
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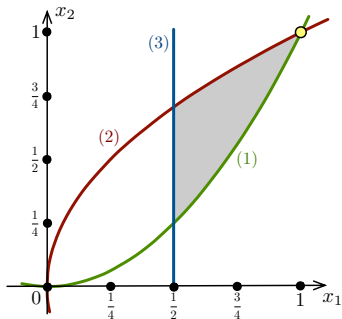
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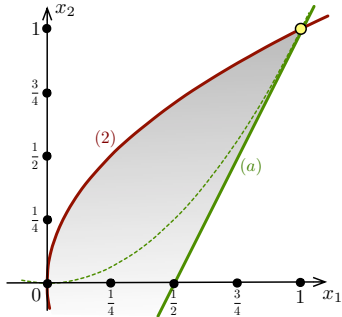
New relaxation

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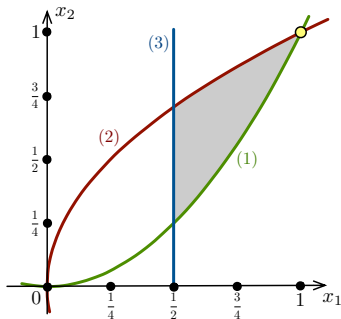
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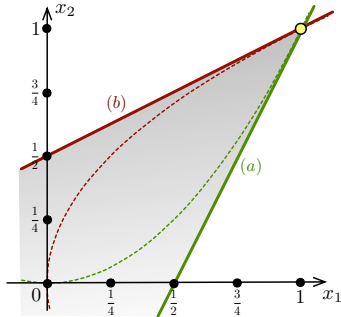
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$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \times \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 1 \times \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \checkmark$$

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The key tool we'll use is **subgradients**.

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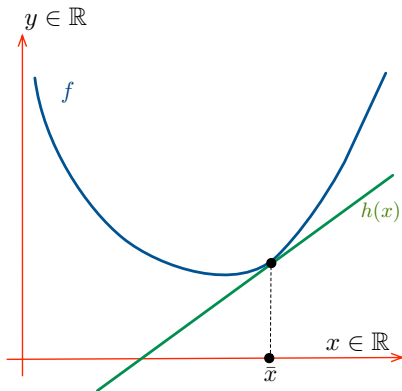
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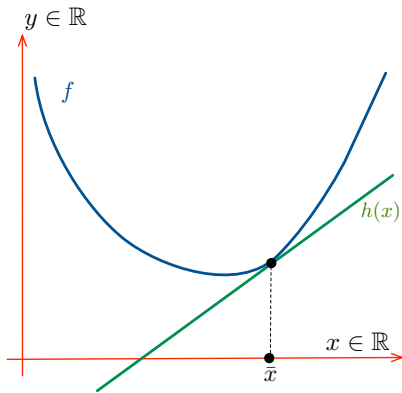
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unique subgradient



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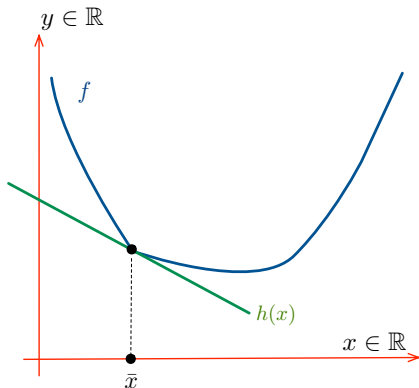
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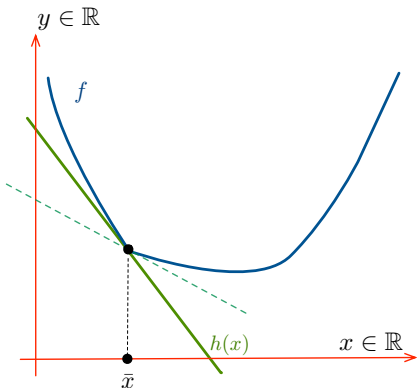
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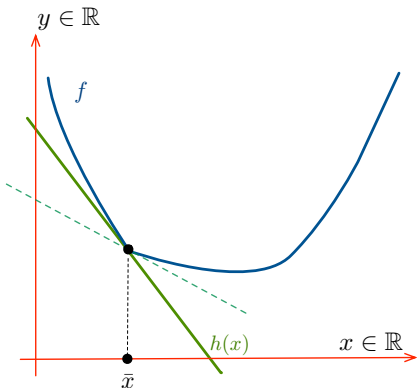
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NOT UNIQUE



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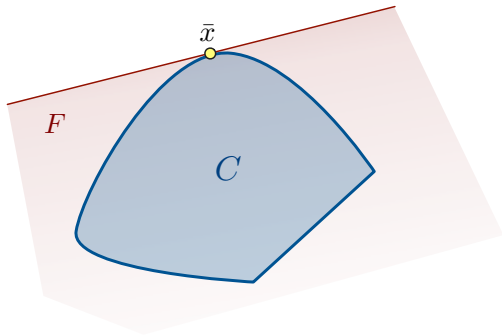
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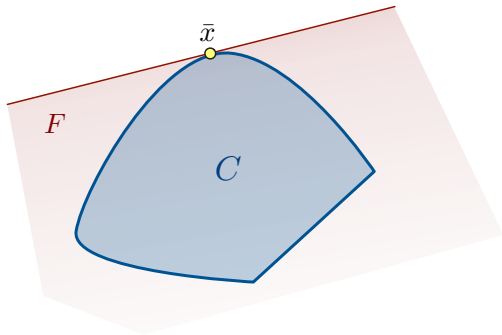


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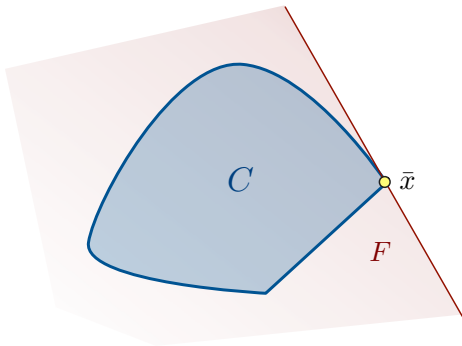
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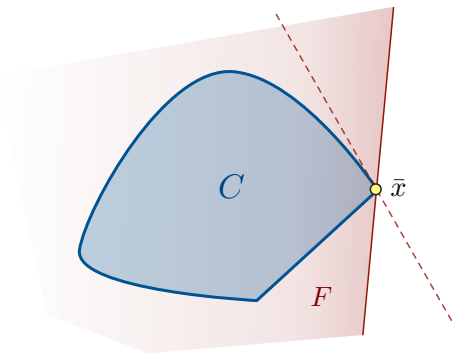


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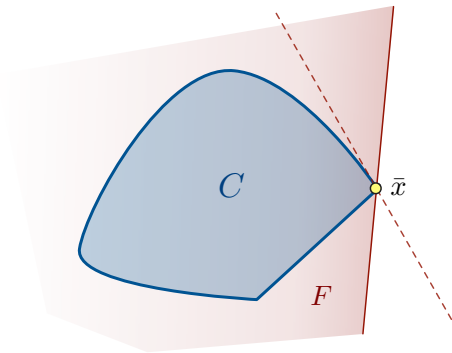


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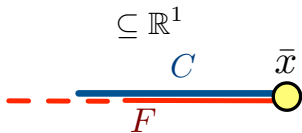
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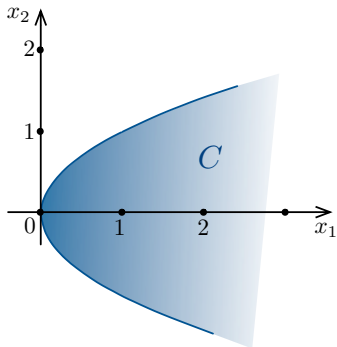
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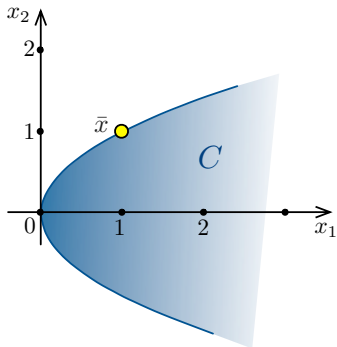
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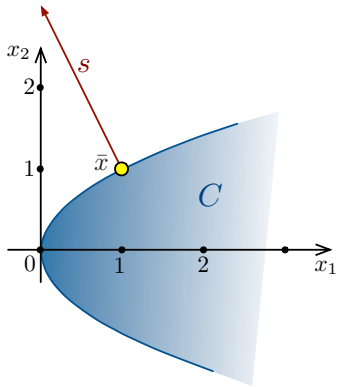
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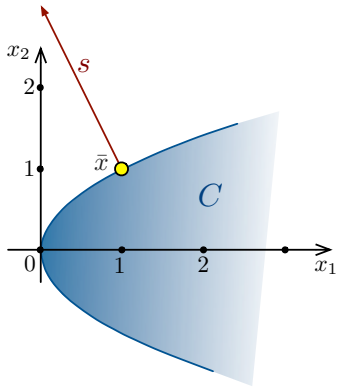
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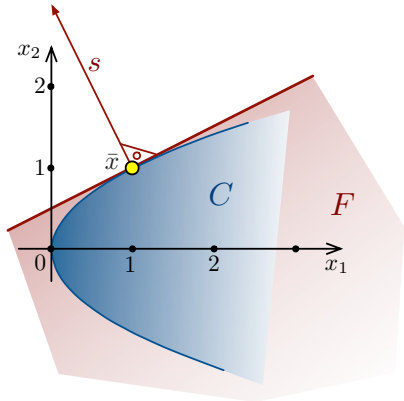
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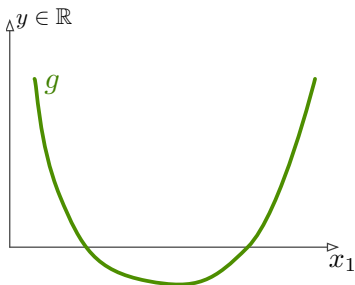
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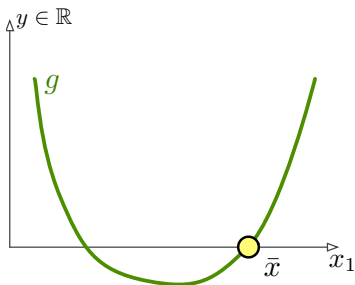
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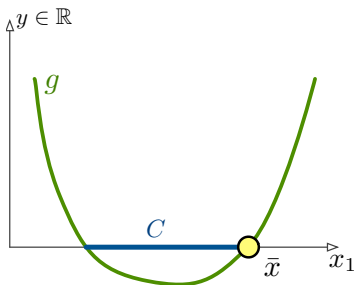
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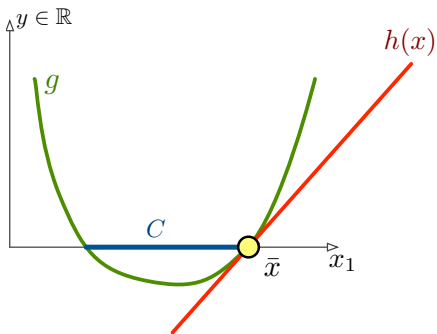
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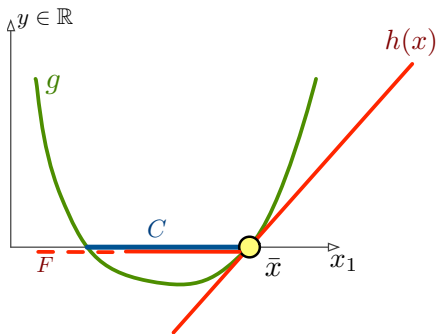
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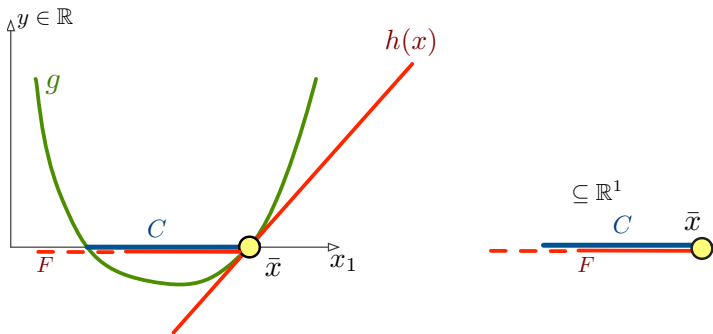
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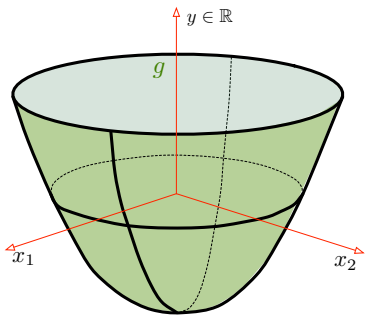
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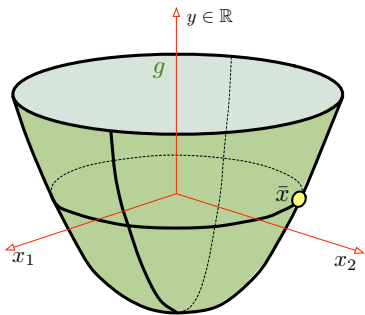
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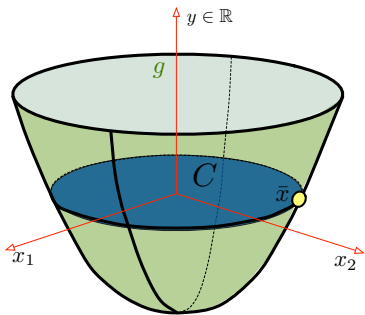
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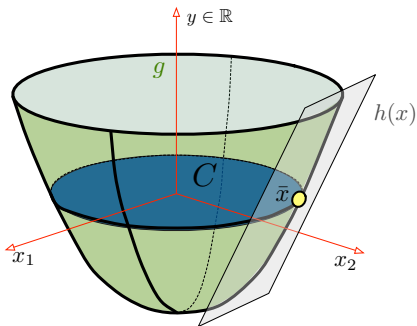
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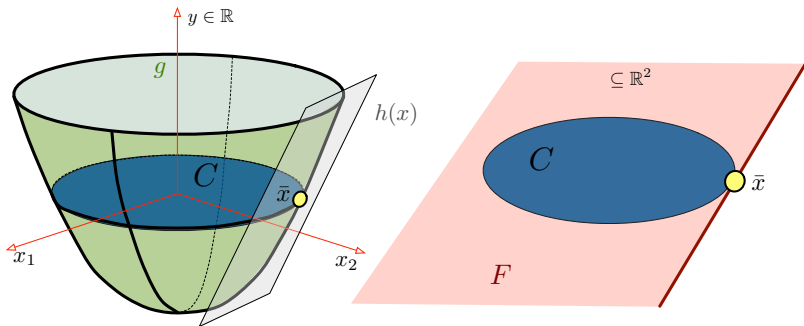
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Why is this relevant for us?

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WE USE IT TO CONSTRUCT RELAXATIONS OF NLPs

$$\min \quad c^\top x$$

s.t.

$$g_i(x) \leq 0 \quad (i = 1, \dots, k)$$

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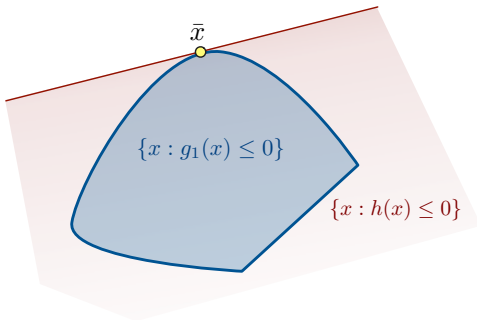
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Example

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$$-x_2 + x_1^2 \leq 0 \quad (1)$$

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$$-\begin{pmatrix} -1 \\ -1 \end{pmatrix} \in \text{cone} \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} \implies \bar{x} \text{ optimal.}$$

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$$g_i(x) \leq 0 \quad (i \in I)$$

We proved that the set of solutions to $g_i(x) \leq 0$

is contained in the set of solutions to $g_i(\bar{x}) + s^{(i)}(x - \bar{x}) \leq 0$.

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$g_i(\bar{x}) + s^{(i)}(x - \bar{x}) \leq 0$ can be rewritten as

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$g_i(\bar{x}) + s^{(i)}(x - \bar{x}) \leq 0$ can be rewritten as

$$s^{(i)}x \leq s^{(i)}\bar{x} - g_i(\bar{x})$$

Proposition

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \\ & g_i(x) \leq 0 \quad (i = 1, \dots, k) \end{array}$$

g_1, \dots, g_k all convex

\bar{x} is a feasible solution

$$\forall i \in I, g_i(\bar{x}) = 0$$

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If $-c \in \text{cone} \{s^{(i)} : i \in I\}$ then \bar{x} is **optimal**.

Proof

We have a relaxation

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This means that \bar{x} is also optimal for the NLP.

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$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \\ & g_i(x) \leq 0 \quad (i = 1, \dots, k) \end{array}$$

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Is there a converse to this result?

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Proposition

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be function and let $\bar{x} \in \mathbb{R}^n$.

If the partial derivative $\frac{\partial f(x)}{\partial x_j}$ exists for f at \bar{x} for all $j = 1, \dots, n$, then the gradient $\nabla f(\bar{x})$ is obtained by evaluating for \bar{x} ,

$$\left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right)^\top.$$

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Compute the gradient of the convex function

$$f(x) = -x_2 + x_1^2$$

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Since $(2, -1)^\top$ is the gradient of f at \bar{x} , it is a subgradient as well.

Definition

A feasible solution to \bar{x} is a **Slater point** of

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & \\ & g_i(x) \leq 0 \quad (i = 1, \dots, k) \end{array}$$

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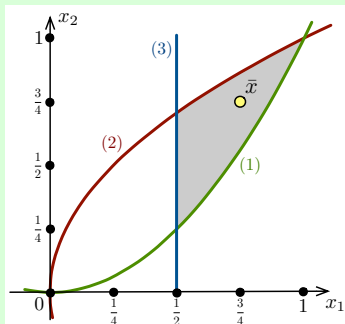
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Remark

We proved the “easy” direction “ \iff ”.

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- We related subgradients and supporting halfspaces.
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- We gave sufficient conditions for a solution to be optimal.
- We stated the KKT theorem.