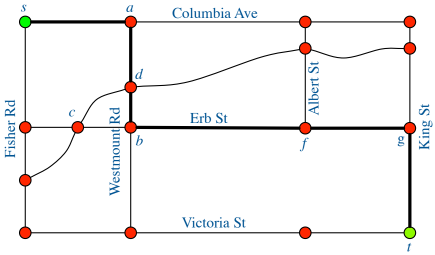
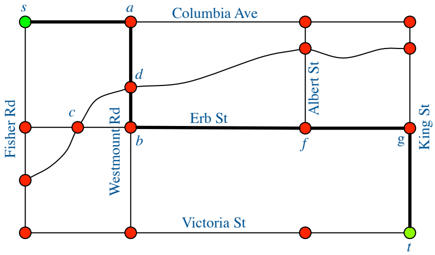
Module 1: Formulations (Optimization on Graphs)

Shortest Paths



- Familiar problem: Starting at location s, we wish to travel to t.
  What is the best (i.e., shortest) route?

Shortest Paths



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- In the figure above, such a route is indicated in bold.

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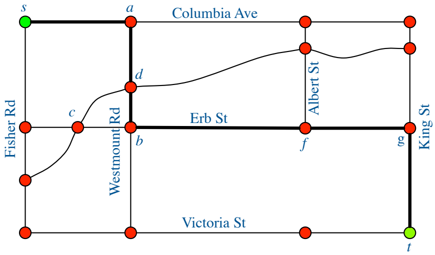

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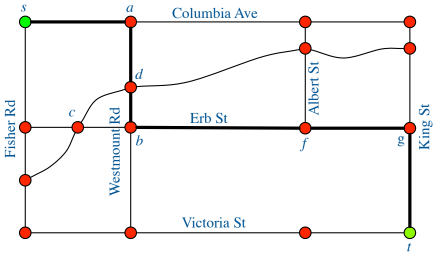

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  ... How?

Graph Theory 101

Rephrasing this problem in the language of graph theory helps!

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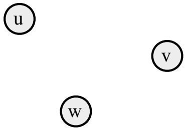

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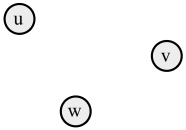

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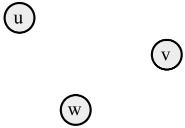

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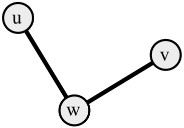

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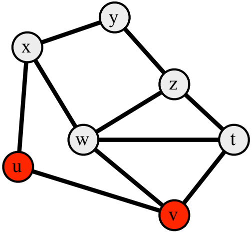

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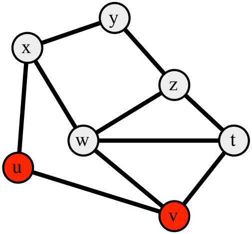

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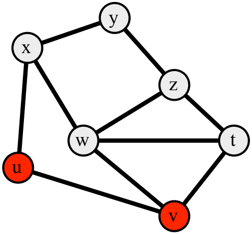

Graphs – Why do we care?

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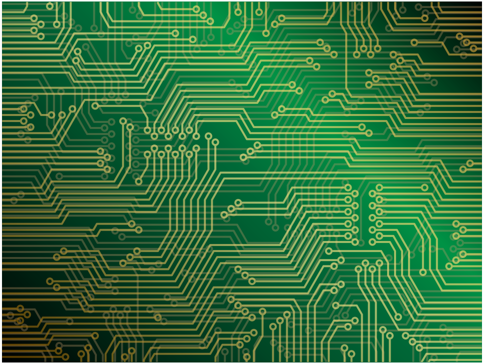

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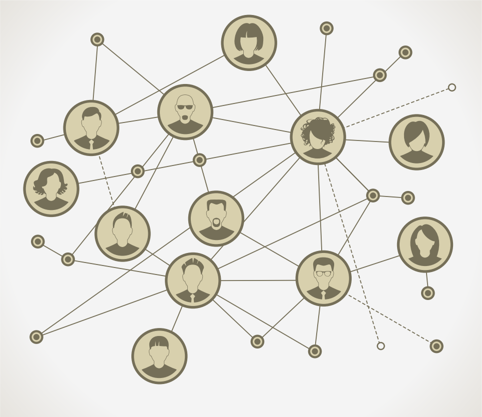

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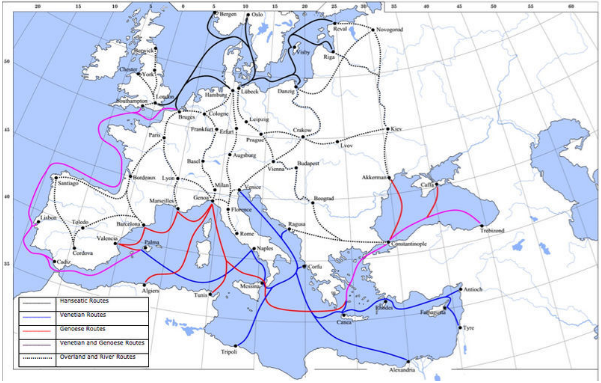

Lampman, 2008 [Online Image]. Late Medieval Trade Routes. Wikimedia Commons.
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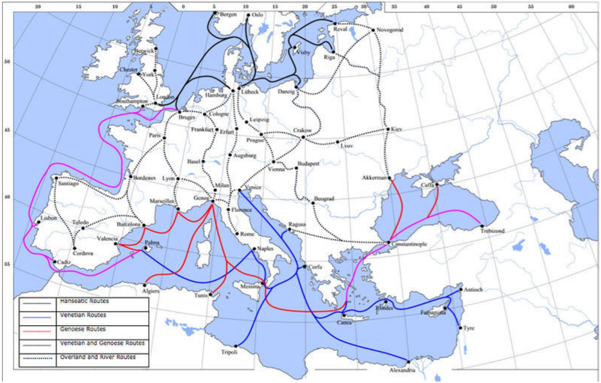
- .... and many more!



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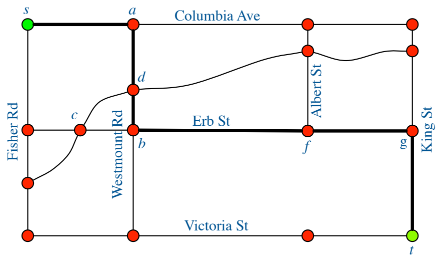
The Map of Water Town



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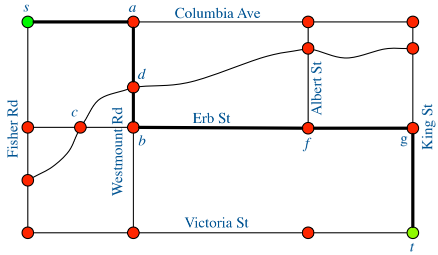

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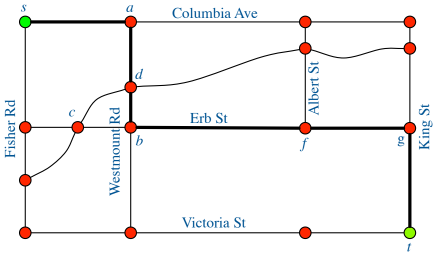

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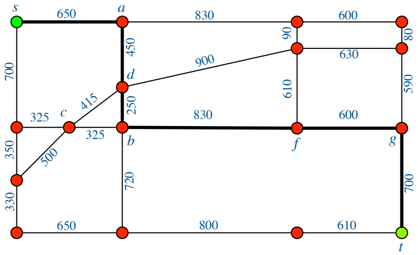

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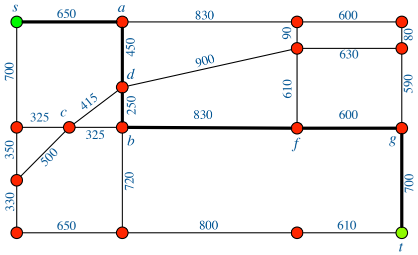

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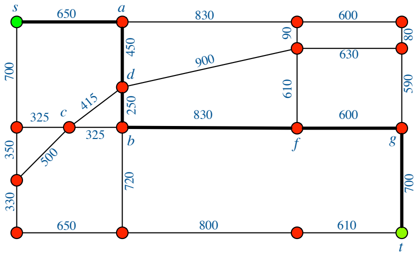
- We are looking for a path connecting s and t of smallest total length!

Paths



An s, t-path in $G = (V, E)$ is a sequence

$$v_1v_2, v_2v_3, v_3v_4, \ldots, v_{k-2}v_{k-1}, v_{k-1}v_k$$

where

- $v_i \in V$ and $v_iv_{i+1} \in E$ for all i, and

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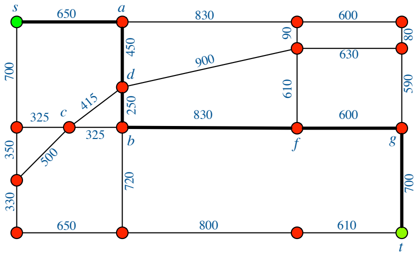

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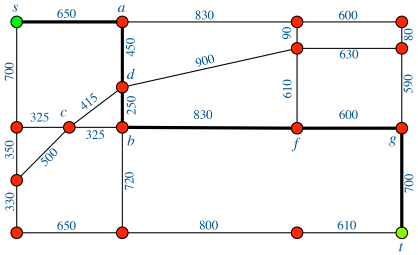

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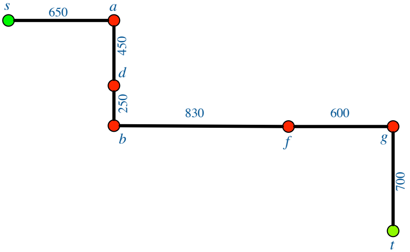

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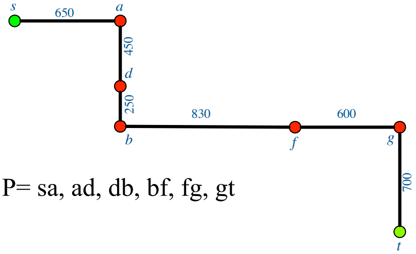

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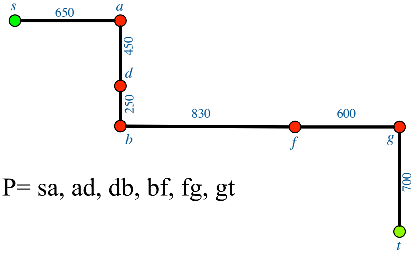

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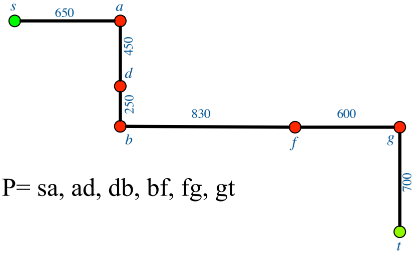

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$$
\begin{aligned}
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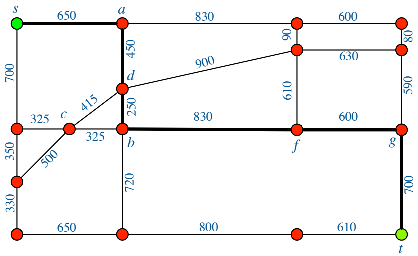

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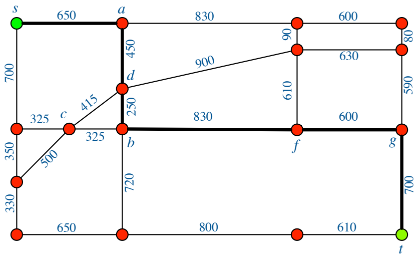

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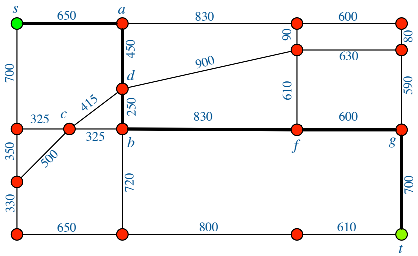

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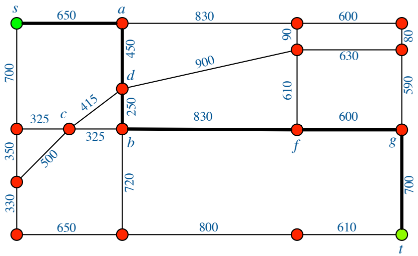

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$\longrightarrow$ Later!

Example: Matchings

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WaterTech has a collection of important jobs:

$$J = \{1', 2', 3', 4'\}$$

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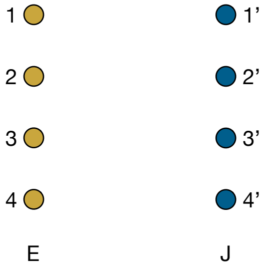

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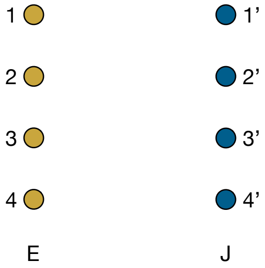

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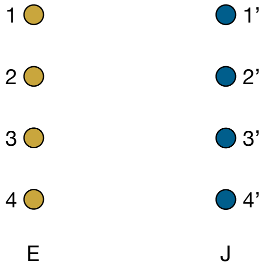

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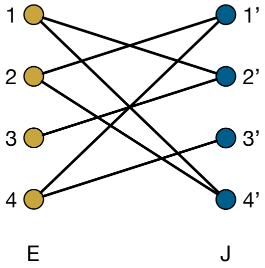

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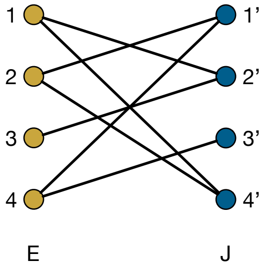

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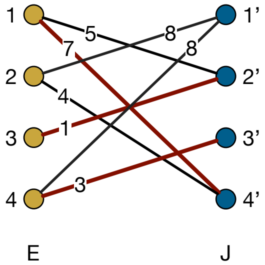

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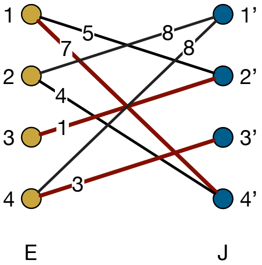

Definition

A collection $M \subseteq E$ is a matching if no two edges $ij, i'j' \in M$ $(ij \neq i'j')$ share an endpoint;

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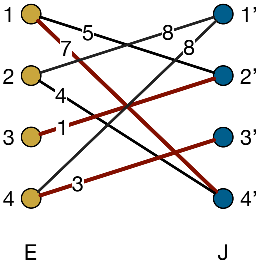

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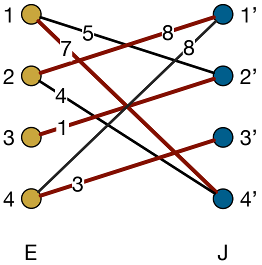

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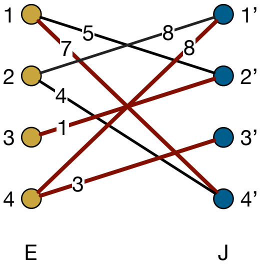

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The cost of a matching M is the sum of costs of its edges:

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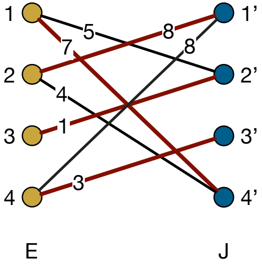

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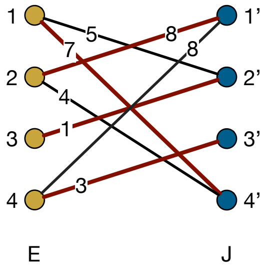

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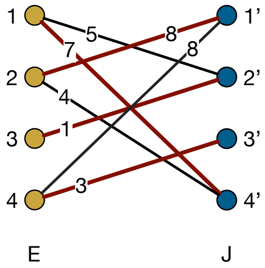

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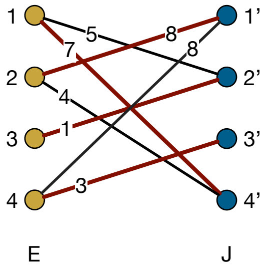

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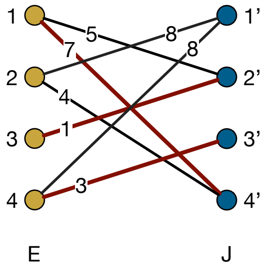

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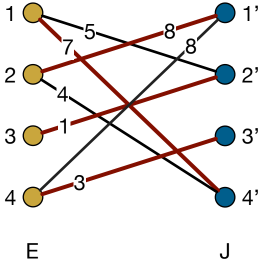
Note: Perfect matchings correspond to feasible assignments of workers to jobs!



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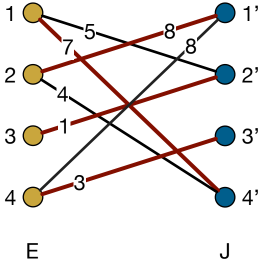

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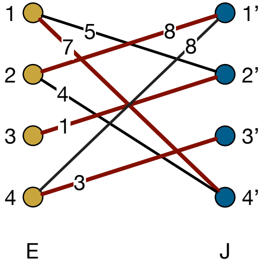

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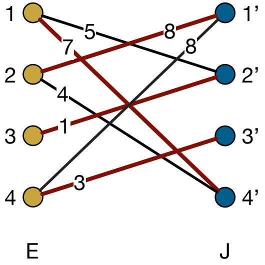


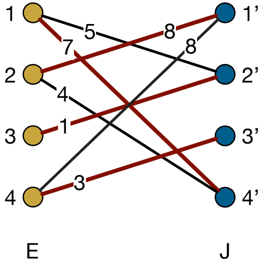
Restatement of original question:

Find a perfect matching M in our graph of smallest cost.

A Little More Notation...

Notation: Use $\delta(v)$ to denote the set of edges incident to v; i.e.,

$$\delta(v) = \{e \in E \,:\, e = vu \text{ for some } u \in V\}.$$



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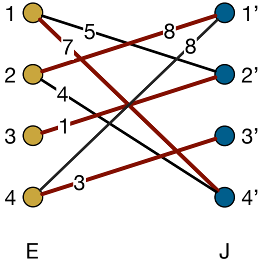

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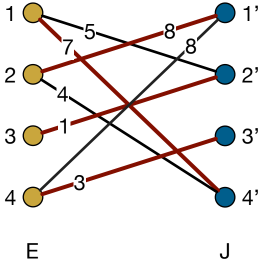

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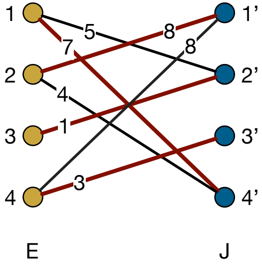

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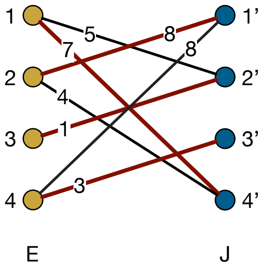

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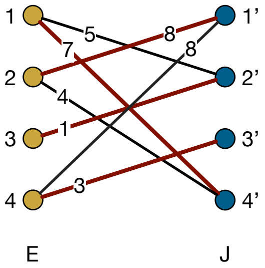

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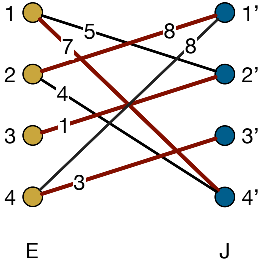

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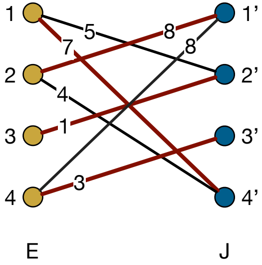

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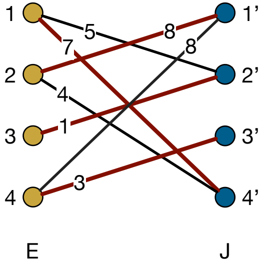

Objective:

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An IP for Perfect Matchings

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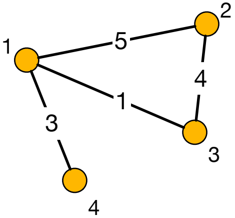

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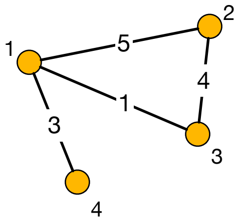

$$\min \quad (5,1,3,4)x$$

$$\text{s.t.} \quad \begin{array}{c c} & \begin{array}{cccc} 12 & 13 & 14 & 23 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} x = \mathbb{1}$$

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