Module 1: Formulations (Nonlinear Models)

So far ...

- Linear programs, and

$$\min \ c^T x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq \mathbb{0}$$

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- Integer linear programs.

Both have linear/affine constraints.

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Now: Nonlinear generalization!

Nonlinear Programs

A non-linear program (NLP) is of the form

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & \dots \\ & g_m(x) \leq 0, \end{aligned}$$

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Note: Linear programs are NLPs!

Example 1: Finding Close Points in an LP

Finding Close Points in an LP

Problem: we are given an LP (P), and an infeasible point $\bar{x}$.

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e.g.: find a point $x \in P$ that minimizes the Euclidean distance to $\bar{x}$:

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Example 2: Binary IP via NLP

NLPs and Binary IPs

Suppose we are given a binary IP (i.e., an integer program all of whose variables are binary).

$$\begin{aligned}
\max \quad & c^T x \\
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& x \ge \mathbb{0} \\
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Recall: (binary) IPs are generally hard to solve!

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$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge \mathbb{0} \\ & x_j(1 - x_j) = 0 \quad (j \in [n]) \qquad (\star) \end{aligned}$$

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Q: Can you change the NLP to express the fact that x_j is any non-negative integer instead of binary?

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Recall: (binary) IPs are generally hard to solve!

Now: can write any binary IP as an NLP!

Ideas?

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge \mathbb{0} \\ & \sin(\pi\, x_j) = 0 \quad (j \in [n]) \qquad (*) \end{aligned}$$

Correctness: For $j \in [n]$, ($\star$) is holds iff $x_j = 0$ or $x_j = 1$.

Q: Can you change the NLP to express the fact that x_j is any non-negative integer instead of binary?

Correctness: note that $\sin(\pi\, x_j) = 0$ only if x_j is an integer.

Example 3: Fermat's Last Theorem

Fermat's Last Theorem

Conjecture [Fermat, 1637]

There are no integers $x,\ y,\ z \geq 1$ and $n \geq 3$ such that

$$x^n + y^n = z^n.$$

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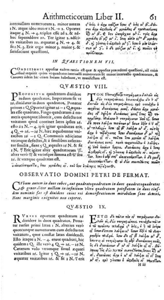

Arithmeticorum Liber II. 61

IN QVAESTIONEM VII.

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

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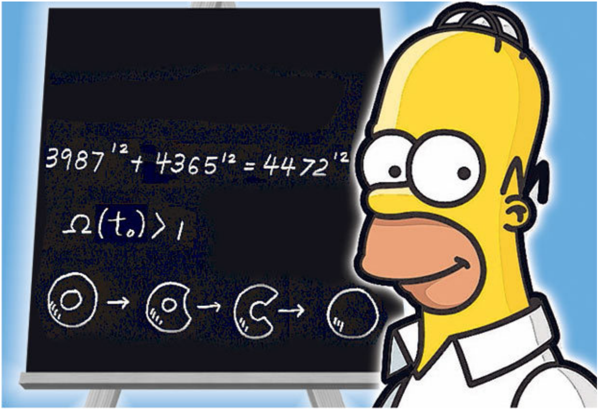

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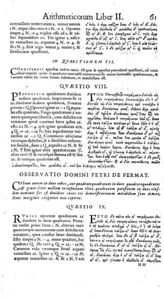

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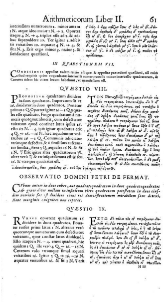

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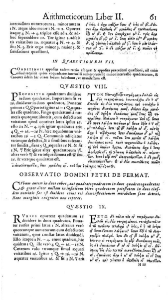

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Some 358 years later, Sir Andrew Wiles gave the first accepted proof of the theorem. The proof is over 150 pages long!

NLP for Fermat's Last Theorem

$$\min \quad (x_1^{x_4} + x_2^{x_4} - x_3^{x_4})^2 + (\sin \pi\, x_1)^2 + (\sin \pi\, x_2)^2 + (\sin \pi\, x_3)^2 + (\sin \pi\, x_4)^2$$

$$\text{s.t.} \quad x_i \geq 1 \quad (i = 1 \ldots 3)$$

$$x_4 \geq 3$$

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Fermat's Last Theorem is true iff the NLP has optimal value greater than 0.

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Proving Fermat's Last Theorem amounts to showing that the value 0 can not be attained!

Recap

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- Some famous questions in Math can easily be reduced to solving certain NLPs