Module 2: Linear Programs (Simplex – A First Attempt)

A Naive Strategy for Solving an LP

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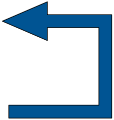

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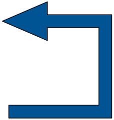

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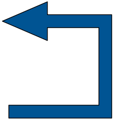

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Questions

- How do we find a feasible solution?

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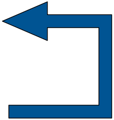

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- How do we find a "better" solution?

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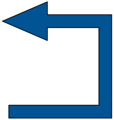

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Questions

- How do we find a feasible solution?
- How do we find a "better" solution?
- Will this ever terminate?

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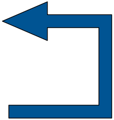

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The SIMPLEX algorithm works along these lines.

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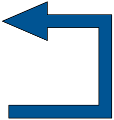

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- How do we find a feasible solution?
- How do we find a "better" solution?
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The SIMPLEX algorithm works along these lines.

In this lecture: A first attempt at this algorithm.

A First Example

A First Example

Consider

$$\begin{aligned} \max \quad & (4,3,0,0)x + 7 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

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Remarks

- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.

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- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.
- The objective function is $z = 4x_1 + 3x_2 + 7$.

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The feasible solution has objective value: $4 \times 0 + 3 \times 0 + 7 = 7$.

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Question

The feasible solution has objective value: $4 \times 0 + 3 \times 0 + 7 = 7$.

- Can we find a feasible solution with value larger than 7?

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Question

The feasible solution has objective value: $4 \times 0 + 3 \times 0 + 7 = 7$.

- Can we find a feasible solution with value larger than 7?

YES!

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Idea

Increase x_1 as much as possible, and keep x_2 unchanged, i.e.,

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- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.
- The objective function is $z = 4x_1 + 3x_2 + 7$.

Idea

Increase x_1 as much as possible, and keep x_2 unchanged, i.e.,

$$\begin{aligned} x_1 &= t && \text{for some } t \geq 0 \text{ as large as possible} \\ x_2 &= 0 \end{aligned}$$

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

Choose $t \geq 0$ as large as possible.

$$\max \quad (4,3,0,0)x + 7$$

s.t.

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Choose $t \geq 0$ as large as possible.

It needs to satisfy

1. the equality constraints, and

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$$x_1 = t$$
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Choose $t \geq 0$ as large as possible.

It needs to satisfy

1. the equality constraints, and
2. the non-negativity constraints.

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = t$$
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$$x_3 = ?$$
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$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x$$

Satisfying the Equality Constraints

$$\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = t$$
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$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Satisfying the Equality Constraints

$$\max \quad (4,3,0,0)x + 7$$

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&= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}
\end{aligned}$$

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$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

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$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Remark

Equality constraints hold for any choice of t.

Satisfying the Non-Negativity Constraints

$$\max \quad (4,3,0,0)x + 7$$

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Choose $t \geq 0$ as large as possible.

Satisfying the Non-Negativity Constraints

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Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

Satisfying the Non-Negativity Constraints

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Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0$ ⇒ $t \leq \frac{2}{3}$

Satisfying the Non-Negativity Constraints

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$x_3 = 2 - 3t \geq 0$ ➡ $t \leq \frac{2}{3}$

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Satisfying the Non-Negativity Constraints

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Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

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$x_3 = 2 - 3t \geq 0$ ➡ $t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0$ ➡ $t \leq 1$

Thus, the largest possible t is $\min\left\{1, \frac{2}{3}\right\} = \frac{2}{3}$.

Satisfying the Non-Negativity Constraints

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$$x_1 = t$$
$$x_2 = 0$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0$ ➡ $t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0$ ➡ $t \leq 1$

Thus, the largest possible t is $\min\left\{1, \frac{2}{3}\right\} = \frac{2}{3}$. The new solution is

$$x = (t,\ 0,\ 2-3t,\ 1-t)^\top = \left(\frac{2}{3}, 0, 0, \frac{1}{3}\right)^\top$$

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_4 = \frac{1}{3}$$

Repeating the Argument?

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$$x_1 = \frac{2}{3}$$
$$x_2 = 0$$
$$x_3 = 0$$
$$x_4 = \frac{1}{3}$$

Question

Is the new solution optimal?

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

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$$x_1 = \frac{2}{3}$$
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Question

Is the new solution optimal? NO!

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

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$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$
$$x_2 = 0$$
$$x_3 = 0$$
$$x_4 = \frac{1}{3}$$

Question

Is the new solution optimal? NO!

Question

Can we use the same trick to get a better solution?

Repeating the Argument?

$$\begin{aligned} \max \quad & (4,3,0,0)x + 7 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$x_1 = \frac{2}{3}$$
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Question

Is the new solution optimal? NO!

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What made it work the first time around?

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The LP needs to be in “canonical” form.

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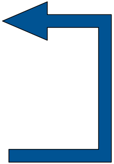

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- Define canonical forms.