Assignment 8

Discussed during the tutorial on December 15th, 2022

8.1 (10 points) A *vertex cover* of a graph G is a set S of vertices of G such that each edge of G is incident with at least one vertex of S. The following IP finds a vertex cover of minimum cardinality:

$$\min \sum_{e \in E(G)} x_e \text{ such that} x_i + x_j \geq 1, e \in E; x \geq 0 \ .$$

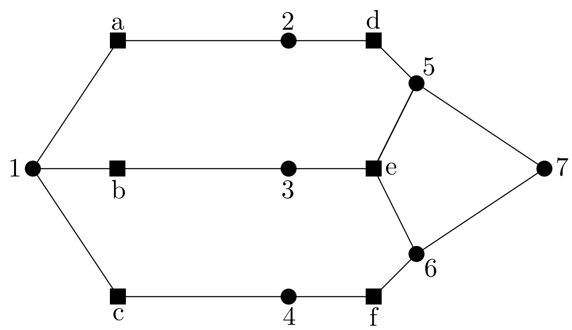
Denote by (P) the LP relaxation of this IP, where the integrality of x is no longer required.

(a) Find the dual (D) of (P).

(b) Show that the largest size (number of edges) of a matching in G is a lower bound on the size of a minimum vertex cover of G.
Hint: use part (a).

(c) Give an example where all matchings have size strictly less than the optimal solutions of (P) and (D) and where these are strictly less than the size of a minimum vertex cover.

8.2 (10 points) The following graph has vertices of two types, a set $H = \{1, \ldots, 7\}$ of hubs, indicated by filled circles, and a set $C = \{a, b, c, d, e, f\}$ of connectors, indicated by squares. A subset S of the hubs is dominant of each connector in C has an edge to at least one hub in S. For instance, $S = \{1, 3, 7\}$ is dominant because a, b, and c have edges to 1, d and f have edges to 7, and e has an edge to 3.



(a) Formulate as an IP the problem of finding a dominant set $S \subseteq H$ which is as small as possible.
Hint: Assign a binary variable to each hub and a constraint for each connector.

Denote by (P) the LP relaxation of the IP given in (a).

(b) State the dual (D) of (P).

(c) Find a solution of (D) of value greater than 2.

(d) Prove that every dominant set of hubs contains at least three hubs.

Assignment continues on the next page.

8.3 (10 points) Consider the following algorithm to find a shortest st-path in a graph G:

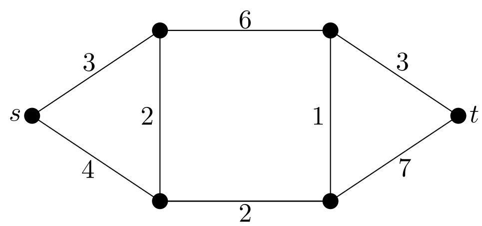
Input: A graph G with edge costs $c_e \geq 0$ for all $e \in E(G)$, distinct vertices $s, t \in V(G)$.

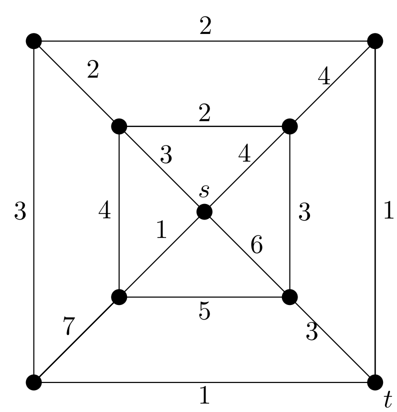
(a) Assume that G has at least one st-path.

Output: A shortest st-path P.

1. Set $y_W = 0$ for all st-cuts $\delta(W)$.
2. Set $U = \{s\}$.
3. Set $\mathsf{slack}(e) = c_e - \sum_{\delta(U)\ st-\text{cut containing } e} y_U$ for each $e \in E(G)$.
4. **while** $t \notin U$ **do**
5. Let $\{a, b\}$ be an edge in $\delta(U)$ of smallest slack for y, where $a \in U$,$b \notin U$.
6. Set $y_U := \mathsf{slack}_y(\{a, b\})$.
7. Set $U := U \cup b$.
8. Change the edge $\{a, b\}$ into the arc $\overrightarrow{ab}$.
9. **end while**
10. **return** a directed st-path P.

For each of the following two graphs, find a shortest path between s and t using the algorithm just given:





In the figures above, each edge is labeled by its length. Make sure to describe for each step of the algorithm, which edge becomes an arc, which vertex is added to the set U, and which st-cut is assigned a positive width. At the end of the procedure, give the shortest st-path and certify that it is indeed a shortest st-path by exhibiting feasible widths.

8.4 (10 points) Let G be a graph with distinct vertices s and t, and non-negative edge weights c.

(a) Show that if G has no st-path, then the following LP is unbounded:

$$\max \sum_{\delta(U) \text{ is } st\text{-cut}} y_U \text{ s.t.} \sum_{\delta(U) \text{ is } st\text{-cut containing } e} y_U \leq c_e, e \in E(G); y_U \geq 0, \delta(U) \text{ is } st\text{-cut} \ .$$

(b) Show that the following ILP has an optimal solution, if and only if G has an st-path:

$$\min \sum_{e \in E(G)} c_e x_e \text{ subject to} \sum_{e \in \delta(U)} x_e \geq 1, U \subseteq V(G), s \in U, t \notin U; x_e \in \mathbb{Z}_{\geq 0}, e \in E(G) \ .$$

Upload your solutions as a `.pdf`-file to the course page on the TUHH e-learning portal until 8am on December, 13th.