Module 1: Formulations (LP Models)

Constrained Optimization

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This class: all functions are affine.

Modeling: Linear Programs

Affine Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = a^T x + \beta$ for $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$.

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(ii) $f(x) = -3x_1 + 5x_3$ (linear)

(iii) $f(x) = 5x - 3\cos(x) + \sqrt{x}$ (not affine and not linear)

Definition

The optimization problem

$$\min\{f(x) : g_i(x) \leq b_i, \\ \forall 1 \leq i \leq m, x \in \mathbb{R}^n\} \tag{P}$$

is called a linear program if f is affine and $g_1, \ldots, g_m$ is finite number of linear functions.

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Instead of set notation, we often write LPs more verbosely

$$\begin{aligned} \max \quad & -2x_1 + 7x_3 \\ \text{subject to} \quad & x_1 + 7x_2 \leq 3 \\ & 3x_2 + 4x_3 \leq 2 \\ & x_1, x_3 \geq 0 \end{aligned}$$

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We often write $x \geq \mathbb{0}$ as a short for all variables are non-negative.

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Second mathematical program is not an LP. Three reasons:

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$$\begin{aligned} \max \quad & -1/x_1 - x_3 \\ \text{subject to} \quad & 2x_1 + x_3 < 3 \\ & x_1 + \alpha\, x_2 = 2 \quad \forall \alpha \in \mathbb{R} \end{aligned}$$

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Production revisited

$$
\begin{array}{llcr}
\max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u & & \\
& 11x_1 + 7x_2 + 6x_3 + 5x_4 & \leq & 700 \\
& 4x_1 + 6x_2 + 5x_3 + 4x_4 & \leq & 500 \\
& 8x_1 + 5x_2 + 5x_3 + 6x_4 & \leq & y_s \\
\text{s.t.} & 7x_1 + 8x_2 + 7x_3 + 4x_4 & \leq & y_u \\
& y_s & \leq & 600 \\
& y_u & \leq & 650 \\
& x_1, x_2, x_3, x_4, y_u, y_s & \geq & 0.
\end{array}
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The mathematical program for WaterTech example from last class is in fact an LP!

Example: Multiperiod Models

Main feature of WaterTech production model:
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In practice, we often have to make a sequence of decision that influence each other.

One such example: Multiperiod Models:

- Time is split into periods,
- We have to make a decision in each period, and
- All decisions influence the final outcome.

KW Oil

KW Oil is local supplier of heating oil

Needs to decide on how much oil to purchase in order to satisfy demand of its customers

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Question: When should we purchase how much oil?

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Additional complication: Company has storage tank that

- Currently (beginning of month 1) contains 2000 litres of oil, and
- Has a capacity of 4000 litres of oil.

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Additional complication: Company has storage tank that

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- Has a capacity of 4000 litres of oil.

Assumption: Oil is delivered at beginning of month, and consumption occurs mid month

KW Oil Model – Variables

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

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$\longrightarrow$ variable t_i for $i \in [4]$

Objective function

Minimize cost of oil procurement.

Variables:

p_i : oil purchase in month i
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$$\min \quad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

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Objective function

Minimize cost of oil procurement.

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Constraints: when do

$$t_1, \ldots, t_4, p_1, \ldots, p_4$$

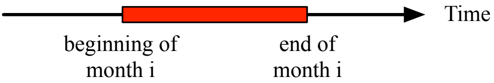
correspond to a feasible purchasing scheme?

Variables:

p_i : oil purchase in month i
t_i : tank level in month i

Constraints

Assumptions:



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Assumptions:

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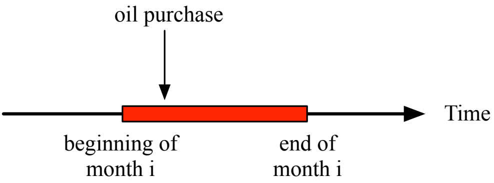

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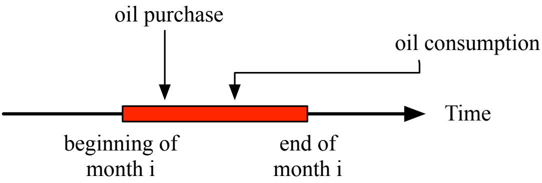

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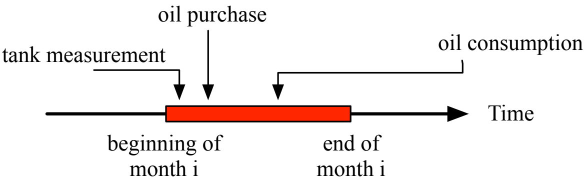

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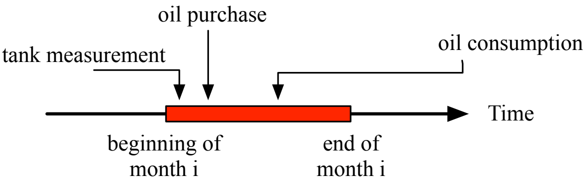

Variables:

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We need: $p_i + t_i \geq \{\text{demand in month } i\}$

Constraints

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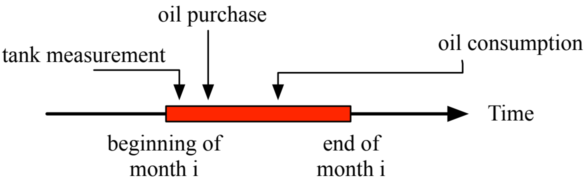

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[Balance Equation] $p_i + t_i = \{\text{demand in month } i\} + t_{i+1}$

Constraints

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Tank content in month 1: 2000 litres

Constraints

[Balance Equation] $p_i + t_i =$ {demand in month i} $+t_{i+1}$

Month 1:
$p_1 + 2000 = 5000 + t_2$

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

Tank content in month 1: 2000 litres

Constraints

[Balance Equation] $p_i + t_i = \{\text{demand in month } i\} + t_{i+1}$

Month 1:

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Month 3:

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$p_2 + t_2 = 8000 + t_3$

Month 3:

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Month 4:

$p_4 + t_4 \geq 6000$

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

Tank content in month 1: 2000 litres

KW Oil: Entire LP

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\begin{array}{ll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 \\
\text{subject to} &
\end{array}
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t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2,3,4) \\
t_i, p_i & \geq & 0 & (i = 1,2,3,4)
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Solution: $p = (3000, 12000, 5000, 6000)^T$, and
$t = (2000, 0, 4000, 0)^T$

KW Oil: Add-Ons

Can easily capture additional features. E.g. ...

> Storage comes at a cost: storage cost is \$.15 per litre/month.

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Add $\sum_{i=1}^{4} .15t_i$ to objective.

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(i) We will need a new variable M for maximum #l purchased

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& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2,3,4) \\
& t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array}
$$

KW Oil: Add-Ons

Can easily capture additional features. E.g. ...

> Storage comes at a cost: storage cost is \$.15 per litre/month.
>
> Add $\sum_{i=1}^{4} .15t_i$ to objective.
>
> Minimize maximum #l of oil purchased over all months
>
> (i) We will need a new variable M for maximum #l purchased
>
> (ii) Will have to add constraints

$$
\begin{array}{lllll}
\text{min} & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2,3,4) \\
& t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array}
$$

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

$$\min \quad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

subject to

$$\begin{array}{lcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2,3,4) \\
t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array}$$

Goal: Minimize maximum #l of oil purchased over all months.

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

(ii) Add constraints

$$p_i \leq M$$

for all $i \in [4]$.

$$\begin{array}{lllll}
\text{min} & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2,3,4) \\
& t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array}$$

Goal: Minimize maximum #l of oil purchased over all months.

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

(ii) Add constraints

$$p_i \leq M$$

for all $i \in [4]$.

(iii) Done?

$$\begin{array}{lllll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2,3,4) \\
& t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array}$$

Goal: Minimize maximum #l of oil purchased over all months.

KW Oil: Add-Ons

(i) Add variable M for maximum #l purchased over all months.

(ii) Add constraints

$$p_i \leq M$$

for all $i \in [4]$.

(iii) Done? No! Need to replace objective function by

$$\min \quad M$$

$$
\begin{array}{lllll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
 & p_1 + t_1 & = & 5000 + t_2 & \\
 & p_2 + t_2 & = & 8000 + t_3 & \\
 & p_3 + t_3 & = & 9000 + t_4 & \\
 & p_4 + t_4 & \geq & 6000 & \\
 & t_1 & = & 2000 & \\
 & t_i & \leq & 4000 & (i = 2, 3, 4) \\
 & t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
$$

Goal: Minimize maximum #l of oil purchased over all months.

Minimizing the Maximum Purchase: LP

$$\min \quad M$$

s.t.

$$\begin{array}{lcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
p_i & \leq & M & (i = 1, 2, 3, 4) \\
t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}$$

KW Oil: Correctness

Why is this a correct model?

$$
\begin{aligned}
\min \quad & M \\
\text{s.t.} \quad & \\
& \begin{array}{rcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
p_i & \leq & M & (i = 1, 2, 3, 4) \\
t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
\end{aligned}
$$

KW Oil: Correctness

Why is this a correct model?

Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

$$
\begin{array}{lllll}
\min & M & & & \\
\text{s.t.} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2, 3, 4) \\
& p_i & \leq & M & (i = 1, 2, 3, 4) \\
& t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
$$

KW Oil: Correctness

Why is this a correct model?

Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

Clearly: $M \geq \max_i p_i$

$$
\begin{array}{llll}
\min & M & & \\
\text{s.t.} & & & \\
& p_1 + t_1 & = & 5000 + t_2 \\
& p_2 + t_2 & = & 8000 + t_3 \\
& p_3 + t_3 & = & 9000 + t_4 \\
& p_4 + t_4 & \geq & 6000 \\
& t_1 & = & 2000 \\
& t_i & \leq & 4000 \qquad (i = 2, 3, 4) \\
& p_i & \leq & M \qquad (i = 1, 2, 3, 4) \\
& t_i, p_i & \geq & 0 \qquad (i = 1, 2, 3, 4)
\end{array}
$$

KW Oil: Correctness

Why is this a correct model?

Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP

Clearly: $M \geq \max_i p_i$

Since M, p, t is optimal we must have $M = \max_i p_i$. Why?

$$\min \quad M$$

s.t.

$$\begin{array}{rcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
p_i & \leq & M & (i = 1, 2, 3, 4) \\
t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}$$