Module 1: Formulations (Nonlinear Models)

So far ...

- Linear programs, and

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq \mathbb{0} \end{aligned}$$

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- Integer linear programs.

Both have linear/affine constraints.

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Now: Nonlinear generalization!

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq \mathbb{0} \\ & x \text{ integer} \end{aligned}$$

Nonlinear Programs

A non-linear program (NLP) is of the form

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\min \quad & f(x) \\
\text{s.t.} \quad & g_1(x) \leq 0 \\
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Note: Linear programs are NLPs!

Example 1: Finding Close Points in an LP

Finding Close Points in an LP

Problem: we are given an LP (P), and an infeasible point $\bar{x}$.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x \in P \end{aligned}$$

$$P = \{x \,:\, Ax \leq b\}$$

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e.g.: find a point $x \in P$ that minimizes the Euclidean distance to $\bar{x}$:

$$\|x - \bar{x}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - \bar{x}_i)^2}$$

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Example 2: Binary IP via NLP

NLPs and Binary IPs

Suppose we are given a binary IP (i.e., an integer program all of whose variables are binary).

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Recall: (binary) IPs are generally hard to solve!

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Ideas?

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge \mathbb{0} \\ & x_j(1 - x_j) = 0 \quad (j \in [n]) \qquad (\star) \end{aligned}$$

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Correctness: For $j \in [n]$, $(\star)$ is holds iff $x_j = 0$ or $x_j = 1$.

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Q: Can you change the NLP to express the fact that x_j is any non-negative integer instead of binary?

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Recall: (binary) IPs are generally hard to solve!

Now: can write any binary IP as an NLP!

Ideas?

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge \mathbb{0} \\ & \sin(\pi\, x_j) = 0 \quad (j \in [n]) \qquad (*) \end{aligned}$$

Correctness: For $j \in [n]$, $(\star)$ is holds iff $x_j = 0$ or $x_j = 1$.

Q: Can you change the NLP to express the fact that x_j is any non-negative integer instead of binary?

Correctness: note that $\sin(\pi\, x_j) = 0$ only if x_j is an integer.

Example 3: Fermat's Last Theorem

Fermat's Last Theorem

Conjecture [Fermat, 1637]

There are no integers x, y, $z \geq 1$ and $n \geq 3$ such that

$$x^n + y^n = z^n.$$

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Arithmeticorum Liber II. 61

interuallum numerorum 2. minor autem 1 N. atque ideo maior 1 N. + 2. Oportet itaque 4 N. + 4. triplos esse ad 2. & adhuc superaddere 10. Ter igitur 2. adscitis vnitatibus 10. æquatur 4 N. + 4. & fit 1 N. 3. Erit ergo minor 3. maior 5. & satisfaciunt quæstioni.

IN QVAESTIONEM VII.

CONDITIONIS appositæ eadem ratio est quæ & appositæ præcedenti quæstioni, nil enim aliud requirit quàm vt quadratus interualli numerorum sit minor interuallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quod continet latus ipsius 16. ...

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. ...

This is false ...

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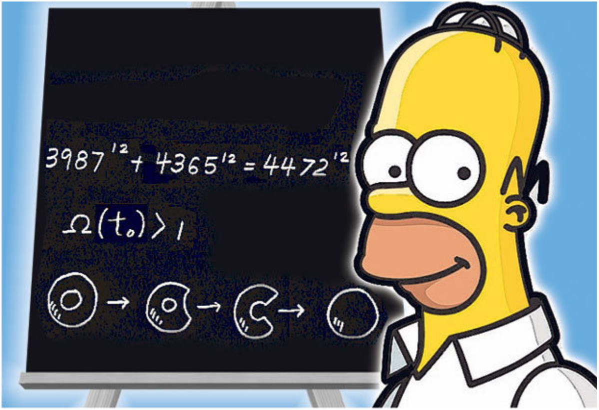

... doh!

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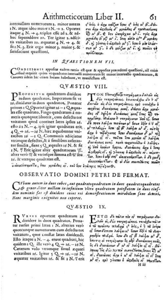

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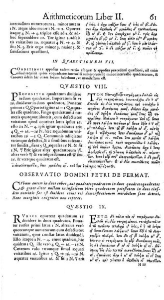

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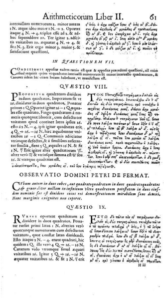

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NLP for Fermat's Last Theorem

$$\begin{aligned} \min \quad & (x_1^{x_4} + x_2^{x_4} - x_3^{x_4})^2 \\ & + (\sin \pi\, x_1)^2 + (\sin \pi\, x_2)^2 + (\sin \pi\, x_3)^2 + (\sin \pi\, x_4)^2 \\ \text{s.t.} \quad & x_i \geq 1 \quad (i = 1 \ldots 3) \\ & x_4 \geq 3 \end{aligned}$$

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    - $\sin \pi\, x_i = 0$, for all $i = 1 \dots 3$.

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Remark

Fermat's Last Theorem is true iff the NLP has optimal value greater than 0.

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Note: well known that there is an infinite sequence of feasible solutions whose objective value converges to 0!

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Proving Fermat's Last Theorem amounts to showing that the value 0 can not be attained!

Recap

- Non-linear programs are of the form

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- Some famous questions in Math can easily be reduced to solving certain NLPs