Module 2: Linear Programs (Certificates)

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

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Consider a linear program.

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Consider a linear program.

- If it is infeasible, how can we prove it?

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- If we have an optimal solution, how can we prove it is optimal?

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Consider a linear program.

- If it is infeasible, how can we prove it?
- If we have an optimal solution, how can we prove it is optimal?
- If it is unbounded, how can we prove it?

This can be always be done!

Proving Infeasibility

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The following linear program is infeasible:

$$\max \quad (3,4,-1,2)^T x$$

s.t.

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

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Question

How can we prove this problem is, in fact, infeasible?

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Question

How can we prove this problem is, in fact, infeasible?

We cannot try all possible assignments of values to x_1, x_2, x_3, and x_4.

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

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Proof

Construct a new equation:

$$\begin{array}{rrrrrrrrl} & -1 \times (1): & (-3 & 2 & 6 & -7)x & = & -6 & \\ + & 2 \times (2): & (4 & -2 & -4 & 8)x & = & 4 & \\ \hline & & (1 & 0 & 2 & 1)x & = & -2 & (\star) \end{array}$$

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Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

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Contradiction!

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

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$$(-1 \ \ 2) \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = (-1 \ \ 2) \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$(1 \ \ 0 \ \ 2 \ \ 1) x = -2 \qquad\qquad (\star)$$

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Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

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$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b}$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

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Contradiction.

This suggests the following result...

Proposition

There is no solution to $Ax = b,\ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

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Exercise

Give a proof of this proposition.

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Give a proof of this proposition.

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If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

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YES!!!!!

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If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

YES!!!!!

Farkas' Lemma

If there is no solution to $Ax = b,\ x \geq \mathbb{0}$, then there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Proving Optimality

$$\max \quad z(x) := (-1 - 4 \;\; 0 \;\; 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

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Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
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How can we prove this solution is, in fact, optimal?

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Question

How can we prove this solution is, in fact, optimal?

We cannot try all possible feasible solutions.

$$\begin{array}{ll} \max & z(x) := (-1\ -4\ \ 0\ \ 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

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Claim

- $\bar{x}$ is feasible solution of value 4.

$$\max \quad z(x) := (-1 - 4 \;\; 0 \;\; 0)x + 4$$

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$$\bar{x}_1 = 0$$
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Claim

- $\bar{x}$ is feasible solution of value 4. (easy)

$$\max \quad z(x) := (-1 - 4 \;\; 0 \;\; 0)x + 4$$

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$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Optimal solution:

$\bar{x}_1 = 0$

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Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

$$\begin{array}{ll} \max & z(x) := (-1\ -4\ \ 0\ \ 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

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Proof

Let x' be an arbitrary feasible solution.

$$\begin{array}{ll} \max & z(x) := (-1 \; -4 \;\; 0 \;\; 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

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Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = (-1 \; -4 \;\; 0 \;\; 0)x' + 4$$

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Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1 \; -4 \;\; 0 \;\; 0)}_{\leq \mathbb{0}} \underbrace{x'}_{\geq \mathbb{0}} + 4$$

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Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1 \ -4 \ \ 0 \ \ 0)x'}_{\leq 0} + 4 \leq 4.$$

Proving Unboudedness

$$\max \quad z := (-1 \;\; 0 \;\; 0 \;\; 1)x$$

s.t.

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Proving Unboudedness

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s.t.

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How can we prove that this problem is unbounded?

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Idea

Construct a family of feasible solutions $x(t)$ for all $t \geq 0$ and show that as t goes to infinity, the value of the objective function goes to infinity.

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Exercise

Generalize and prove the following proposition.

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Proposition

The linear program,

$$\max\{c^T x : Ax = b, x \geq \mathbb{0}\}$$

is unbounded if we can find $\bar{x}$ and r such that

$$\bar{x} \geq \mathbb{0}, \quad r \geq \mathbb{0}, \quad A\bar{x} = b, \quad Ar = \mathbb{0} \quad \text{and} \quad c^T r > 0.$$

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1. For linear programs, exactly one of the following holds. It is
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Remark

We have not yet shown you how to find such proofs.