Module 2: Linear Programs (Canonical Forms)

$$\max\left\{c^{\top}x: Ax = b, x \ge 0\right\}$$
(P)

$$\max\left\{c^{\top}x: Ax = b, x \ge 0\right\}$$
(P)

Definition Let *B* be a basis of *A*. Then (P) is in canonical form for *B* if

$$\max\left\{c^{\top}x: Ax = b, x \ge 0\right\}$$
(P)

Definition

Let B be a basis of A. Then (P) is in canonical form for B if (P1) $A_B = I$, and

$$\max\left\{c^{\top}x: Ax = b, x \ge 0\right\}$$
(P)

Definition

Let B be a basis of A. Then (P) is in canonical form for B if (P1) $A_B = I$, and (P2) $c_j = 0$ for all $j \in B$.

$$\max\left\{c^{\top}x: Ax = b, x \ge 0\right\}$$
(P)

Definition

Let B be a basis of A. Then (P) is in canonical form for B if (P1) $A_B = I$, and (P2) $c_j = 0$ for all $j \in B$.

$$\begin{array}{ll} \max & (0 & 0 & 2 & 4)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

Canonical form for $B = \{1, 2\}$

$$\max\left\{c^{\top}x: Ax = b, x \ge 0\right\}$$
(P)

Definition

Let B be a basis of A. Then (P) is in canonical form for B if (P1) $A_B = I$, and (P2) $c_j = 0$ for all $j \in B$.

$$\begin{array}{ll} \max & (-2 & 0 & 0 & 6)x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

Canonical form for $B = \{2, 3\}$

$$\max\left\{c^{\top}x: Ax = b, x \ge 0\right\}$$
(P)

$$\max\left\{c^{\top}x : Ax = b, x \ge 0\right\}$$
(P)

Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

$$\max\left\{c^{\top}x : Ax = b, x \ge 0\right\}$$
(P)

Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

More formally, we will show the following:

$$\max\left\{c^{\top}x : Ax = b, x \ge 0\right\}$$
(P)

Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

More formally, we will show the following:

Proposition

For any basis B, there exists (P') in canonical form for B such that

$$\max\left\{c^{\top}x : Ax = b, x \ge 0\right\}$$
(P)

Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

More formally, we will show the following:

Proposition

For any basis B, there exists (P') in canonical form for B such that (1) (P) and (P') have the same feasible region, and

$$\max\left\{c^{\top}x : Ax = b, x \ge 0\right\}$$
(P)

Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

More formally, we will show the following:

Proposition

For any basis B, there exists (P') in canonical form for B such that

- (1) (P) and (P') have the same feasible region, and
- (2) feasible solutions have the same objective value for (P) and (P').

For any basis B, there exists (P') in canonical form for B such that

(1) (P) and (P') have the same feasible region, and

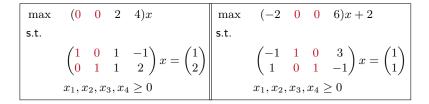
(2) <u>feasible solutions</u> have the same objective value for (P) and (P').

For any basis B, there exists (P') in canonical form for B such that (1) (P) and (P') have the same feasible region, and

(2) <u>feasible solutions</u> have the same objective value for (P) and (P').

$$\begin{array}{ll} \max & (0 & 0 & 2 & 4)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

For any basis B, there exists (P') in canonical form for B such that (1) (P) and (P') have the same feasible region, and (2) <u>feasible solutions</u> have the same objective value for (P) and (P').



For any basis B, there exists (P') in canonical form for B such that (1) (P) and (P') have the same feasible region, and (2) <u>feasible solutions</u> have the same objective value for (P) and (P').

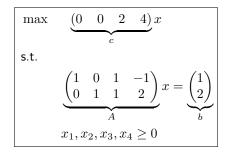
Example

(1) $\bar{x} = (1, 2, 0, 0)^{\top}$ is feasible for both LPs.

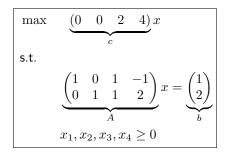
For any basis B, there exists (P') in canonical form for B such that (1) (P) and (P') have the same feasible region, and (2) <u>feasible solutions</u> have the same objective value for (P) and (P').

Example

 $(1) \quad \bar{x} = (1,2,0,0)^\top \text{ is feasible for both LPs}.$



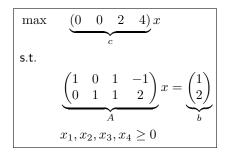
(P)



(P)

Question

How do we rewrite (P) in canonical form for basis $B = \{2, 3\}$?

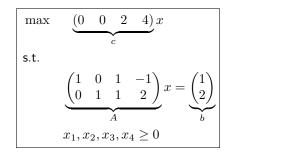


(P)

Question

How do we rewrite (P) in canonical form for basis $B = \{2, 3\}$?

(P1) Replace
$$Ax = b$$
 by $A'x = b'$ with $A'_{\{2,3\}} = I$.



(P)

Question

How do we rewrite (P) in canonical form for basis $B = \{2, 3\}$?

(P1) Replace Ax = b by A'x = b' with $A'_{\{2,3\}} = I$. (P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and \bar{z} is a constant.

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

(P1) Replace Ax = b by A'x = b' with $A'_{\{2,3\}} = I$.

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

(P1) Replace Ax = b by A'x = b' with $A'_{\{2,3\}} = I$. $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

(P1) Replace Ax = b by A'x = b' with $A'_{\{2,3\}} = I$.

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

(P1) Replace Ax = b by A'x = b' with $A'_{\{2,3\}} = I$. $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} -1 & 1 & 0 & 3\\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1\\ 1 \end{pmatrix}$

Consider the system Ax = b with basis B of A.

Consider the system Ax = b with basis B of A.

(P1) Replace Ax = b by A'x = b' with $A'_B = I$ for some basis B.

Consider the system Ax = b with basis B of A.

(P1) Replace Ax = b by A'x = b' with $A'_B = I$ for some basis B.

$$Ax = b$$

Consider the system Ax = b with basis B of A.

(P1) Replace Ax = b by A'x = b' with $A'_B = I$ for some basis B.

Ax = b



$$\underbrace{A_B^{-1}A}_{A'}x = \underbrace{A_B^{-1}b}_{b'}$$

Consider the system Ax = b with basis B of A.

(P1) Replace Ax = b by A'x = b' with $A'_B = I$ for some basis B.

Ax = b

$$\underbrace{A_B^{-1}A}_{A'} x = \underbrace{A_B^{-1}b}_{b'}$$

Remarks

•
$$A'_B = I$$
.

Consider the system Ax = b with basis B of A.

(P1) Replace Ax = b by A'x = b' with $A'_B = I$ for some basis B.

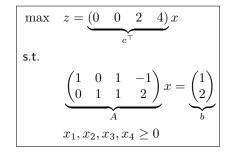
Ax = b

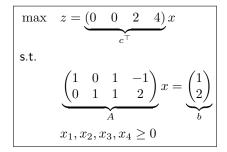
$$\underbrace{A_B^{-1}A}_{A'}x = \underbrace{A_B^{-1}b}_{b'}$$

Remarks

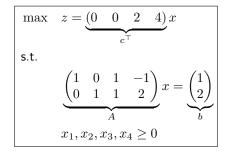
•
$$A'_B = I$$
.

• Ax = b and A'x = b' have the same set of solutions.



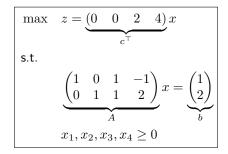


(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and \bar{z} is a constant.



(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and \bar{z} is a constant.

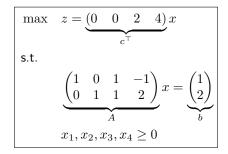
Step 1. Construct a new objective function by



(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and \bar{z} is a constant.

Step 1. Construct a new objective function by

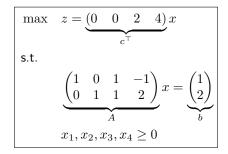
• multiplying constraint 1 by y_1 ,



(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and \bar{z} is a constant.

Step 1. Construct a new objective function by

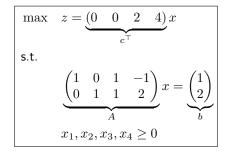
- multiplying constraint 1 by y_1 ,
- multiplying constraint 2 by y_2 , and



(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and \bar{z} is a constant.

Step 1. Construct a new objective function by

- multiplying constraint 1 by y_1 ,
- multiplying constraint 2 by y_2 , and
- adding the resulting constraints to the objective function.



(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ where $\bar{c}_2 = \bar{c}_3 = 0$ and \bar{z} is a constant.

Step 1. Construct a new objective function by

- multiplying constraint 1 by y_1 ,
- multiplying constraint 2 by y_2 , and
- adding the resulting constraints to the objective function.

Step 2. Choose y_1, y_2 to get $\bar{c}_2 = \bar{c}_3 = 0$.

$$\begin{array}{ll} \max & z = (0 & 0 & 2 & 4)x \\ \text{s.t.} & & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{array}{ll} \max & z = (0 & 0 & 2 & 4)x \\ \text{s.t.} & & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

$$(y_1, y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = (y_1, y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{array}{cccc} \max & z = (0 & 0 & 2 & 4)x \\ \text{s.t.} & & \\ & & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & & \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

$$0 = -(y_1 \ y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + (y_1 \ y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{array}{cccc} \max & z = (0 & 0 & 2 & 4)x \\ \text{s.t.} & & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

$$\begin{array}{rcl} 0 & = & & -(y_1 \ y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + (y_1 \ y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ z & = & & (0 \ 0 \ 2 \ 4) x \end{array}$$

0

z

$$\begin{array}{ll} \max & z = (0 & 0 & 2 & 4)x \\ \text{s.t.} & & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

$$0 = -(y_1 \ y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + (y_1 \ y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} z = (0 \ 0 \ 2 \ 4) x$$

$$z = \left[\begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix} - \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \right] x + \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{array}{cccc} \max & z = (0 & 0 & 2 & 4)x\\ \text{s.t.} & & \\ & & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\\ & & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

$$0 = -(y_1 \ y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + (y_1 \ y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} z = (0 \ 0 \ 2 \ 4)x$$

$$z = \left[\begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix} - \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \right] x + \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Remark

For any choice of y_1, y_2 and any feasible solution x,

$$\begin{array}{cccc} \max & z = (0 & 0 & 2 & 4)x \\ \text{s.t.} & & \\ & & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & & \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

$$\begin{array}{rcl} 0 & = & & -(y_1 \ y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + (y_1 \ y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ z & = & & (0 \ 0 \ 2 \ 4) x \end{array}$$

$$z = \left[\begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix} - \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \right] x + \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Remark

For any choice of y_1, y_2 and any feasible solution x,

objective value of x for old objective function = objective value of x for new objective function

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

$$(0 \ 0) = \bar{c}_B^\top = (0 \ 2) - (y_1 \ y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

$$(0 \quad 0) = \bar{c}_B^{\top} = (0 \quad 2) - (y_1 \quad y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$(y_1 \quad y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (0 \quad 2)$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

$$(0 \quad 0) = \bar{c}_B^{\top} = (0 \quad 2) - (y_1 \quad y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$(y_1 \quad y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (0 \quad 2)$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{\top} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

$$(0 \quad 0) = \bar{c}_B^{\top} = (0 \quad 2) - (y_1 \quad y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$(y_1 \quad y_2) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (0 \quad 2)$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{\top} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

How do we choose y_1, y_2 such that $\bar{c}_B = 0$ for $B = \{2, 3\}$?

Choose

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

How do we choose y_1, y_2 such that $\bar{c}_B = 0$ for $B = \{2, 3\}$?

Choose

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$z = \begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (2 \quad 0) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix} x + (2 \quad 0) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$z = \underbrace{\begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix}}_{\bar{c}^{\top}} x + \underbrace{(y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\bar{z}}$$

How do we choose y_1, y_2 such that $\bar{c}_B = 0$ for $B = \{2, 3\}$?

Choose

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$z = \begin{bmatrix} (0 \quad 0 \quad 2 \quad 4) - (2 \quad 0) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{bmatrix} x + (2 \quad 0) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \qquad z = (-2 \quad 0 \quad 0 \quad 6)x + 2$$

$$\begin{array}{ll} \max & z = c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \max & z = c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ with $\bar{c}_B = 0$ (\bar{z} constant) for some basis B.

$$\begin{array}{ll} \max & z = c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ with $\bar{c}_B = 0$ (\bar{z} constant) for some basis B. $y^{\top}Ax = y^{\top}b$

$$\begin{array}{ll} \max & z = c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ with $\bar{c}_B = 0$ (\bar{z} constant) for some basis B. $0 = -y^{\top}Ax + y^{\top}b$

$$\begin{array}{ll} \max & z = c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ with $\bar{c}_B = 0$ (\bar{z} constant) for some basis B. $0 = -y^{\top}Ax + y^{\top}b$ $z = c^{\top}x$

$$\begin{array}{ll} \max & z = c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ with $\bar{c}_B = 0$ (\bar{z} constant) for some basis B.

0	=	$-y^{\top}Ax + y^{\top}b$
z	=	$c^ op x$
z	=	$\left[c^{\top} - y^{\top}A\right]x + y^{\top}b$

$$\begin{array}{cc} \max & z = c^{\top} x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ with $\bar{c}_B = 0$ (\bar{z} constant) for some basis B. $0 = -y^{\top}Ax + y^{\top}b$ $z = c^{\top}x$ $z = [c^{\top} - y^{\top}A]x + y^{\top}b$

Remark

For any choice of y and any feasible solution x,

$$\begin{array}{ll} \max & z = c^{\top} x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0 \end{array}$$

(P2) Replace $c^{\top}x$ by $\bar{c}^{\top}x + \bar{z}$ with $\bar{c}_B = 0$ (\bar{z} constant) for some basis B.

$$0 = -y^{\top}Ax + y^{\top}b$$
$$z = c^{\top}x$$

$$z = \left[c^{\top} - y^{\top}A\right]x + y^{\top}b$$

Remark

For any choice of y and any feasible solution x,

objective value of x for old objective function = objective value of x for new objective function

$$z = \underbrace{\left[\underline{c^{\top} - y^{\top} A} \right]}_{\overline{c}^{\top}} x + \underbrace{y^{\top} b}_{\overline{z}}$$

$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

How do we choose y such that $\bar{c}_B = 0$ for a basis B?

$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

How do we choose y such that $\bar{c}_B = 0$ for a basis B?

$$\mathbb{O}^{\top} = \bar{c}_B^{\top}$$

$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

$$\mathbb{O}^{\top} = \bar{c}_B^{\top} = c_B^{\top} - y^{\top} A_B$$

$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

$$\mathbb{O}^{\top} = \bar{c}_B^{\top} = c_B^{\top} - y^{\top} A_B$$
$$y^{\top} A_B = c_B^{\top}$$



$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

$$0^{\top} = \bar{c}_B^{\top} = c_B^{\top} - y^{\top} A_B$$
$$y^{\top} A_B = c_B^{\top}$$
$$A_B^{\top} y = c_B$$

$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

$$\mathbb{O}^{\top} = \bar{c}_B^{\top} = c_B^{\top} - y^{\top} A_B$$
$$y^{\top} A_B = c_B^{\top}$$
$$A_B^{\top} y = c_B$$
$$y = \left(A_B^{\top}\right)^{-1} c_B$$

$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

How do we choose y such that $\bar{c}_B = 0$ for a basis B?

$$\mathbb{O}^{\top} = \bar{c}_B^{\top} = c_B^{\top} - y^{\top} A_B$$
$$y^{\top} A_B = c_B^{\top}$$
$$A_B^{\top} y = c_B$$
$$y = (A_B^{\top})^{-1} c_B$$

Remark

For any non-singular matrix M,

 $(M^{\top})^{-1} = (M^{-1})^{\top} =: M^{-\top}$

$$z = \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}^{\top}} x + \underbrace{y^{\top}b}_{\bar{z}}$$

How do we choose y such that $\bar{c}_B = 0$ for a basis B?

$$\mathbb{O}^{\top} = \bar{c}_B^{\top} = c_B^{\top} - y^{\top} A_B$$
$$y^{\top} A_B = c_B^{\top}$$
$$A_B^{\top} y = c_B$$
$$y = (A_B^{\top})^{-1} c_B = A_B^{-\top} c_B$$

Remark

For any non-singular matrix M,

 $(M^{\top})^{-1} = (M^{-1})^{\top} =: M^{-\top}$

Proposition

Let B be a basis of A,

$$\begin{array}{c|c} \max & c^{\top}x \\ \text{s.t.} \\ & Ax = b \\ & x \ge 0 \end{array}$$
 (P)

Proposition

Let B be a basis of A,

$$\begin{array}{c} \max \quad c^{\top}x \\ \text{s.t.} \\ Ax = b \\ x \ge 0 \end{array} (\mathsf{P})$$

$$\max \underbrace{\left[c^{\top} - y^{\top} A\right]}_{\bar{c}} x + y^{\top} b$$
s.t.
$$\underbrace{A_B^{-1} A}_{A'} x = A_B^{-1} b$$

$$x \ge 0$$
(P')

Proposition

Let B be a basis of A,

$$\begin{array}{c} \max \quad c^{\top}x \\ \text{s.t.} \\ Ax = b \\ x \ge 0 \end{array}$$
 (P)

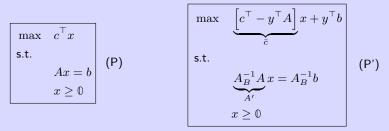
$$\max \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\bar{c}} x + y^{\top}b$$
s.t.
$$\underbrace{A_{B}^{-1}A}_{A'} x = A_{B}^{-1}b$$

$$x \ge 0$$
(P')

where $y = A_B^{-\top} c_B$.

Proposition

Let B be a basis of A,



where $y = A_B^{-\top} c_B$. Then (1) (P') is in canonical form for basis B, i.e., $\bar{c}_B = 0$ and $A'_B = I$.

Proposition

Let B be a basis of A,

$\boxed{\max \ c^{\top}x}$	$\max \underbrace{\left[c^{\top} - y^{\top}A\right]}_{\overline{c}} x + y^{\top}b$	
s.t. $Ax = b$ $x \ge 0$ (P)	s.t. $\underbrace{A_B^{-1}A}_{A'}x = A_B^{-1}b$ $x \ge 0$	(P')

where $y = A_B^{-\top} c_B$. Then

(1) (P') is in canonical form for basis B, i.e., $\bar{c}_B = 0$ and $A'_B = I$. (2) (P) and (P') have the same feasible region.

Proposition

Let B be a basis of A,

$\boxed{\begin{array}{c} \max c^{\top}x \end{array}}$		max	$\underbrace{\left[\boldsymbol{c}^{\top} - \boldsymbol{y}^{\top} \boldsymbol{A} \right]}_{\bar{\boldsymbol{c}}} \boldsymbol{x} + \boldsymbol{y}^{\top} \boldsymbol{b}$	
s.t. $Ax = b$ $x \ge 0$	(P)	s.t.	$\underbrace{A_B^{-1}A}_{A'} x = A_B^{-1}b$ $x \ge 0$	(P')

where $y = A_B^{-\top} c_B$. Then

(1) (P') is in canonical form for basis B, i.e., $\bar{c}_B = 0$ and $A'_B = I$. (2) (P) and (P') have the same feasible region.

(3) <u>Feasible solutions</u> have the same objective value for (P) and (P').