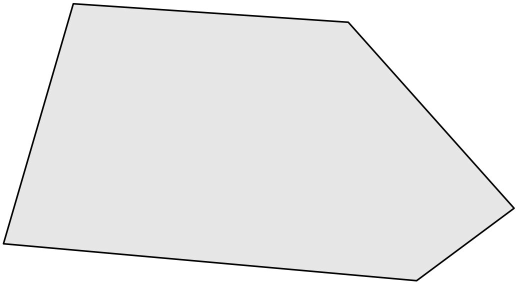
Module 2: Linear Programs (Extreme Points)

Extreme Points

Consider the following convex set:



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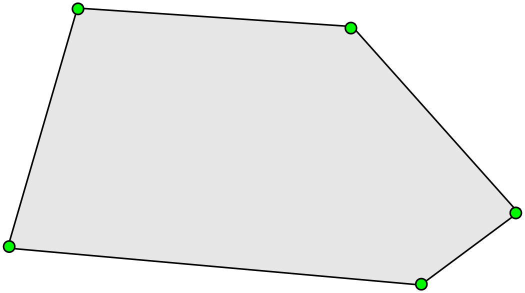

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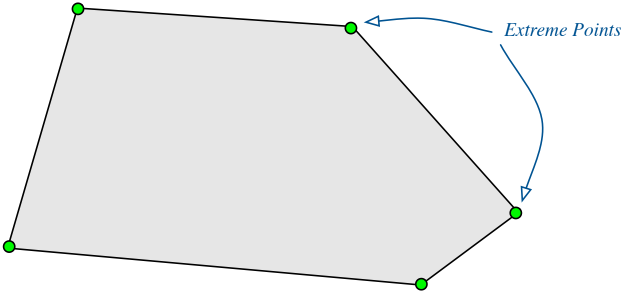

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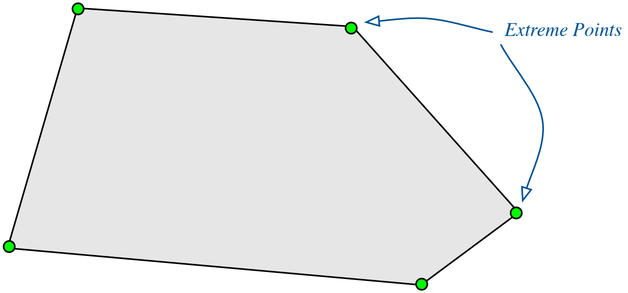

Question

How might we formally describe the "extreme points"?

Towards a Definition of Extreme Points

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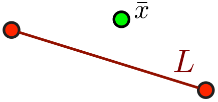

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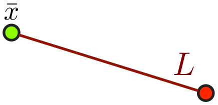

$\bar{x}$ is contained in L,

but NOT properly.

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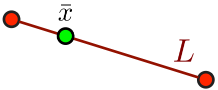

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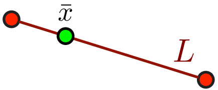

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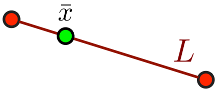

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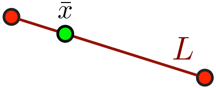

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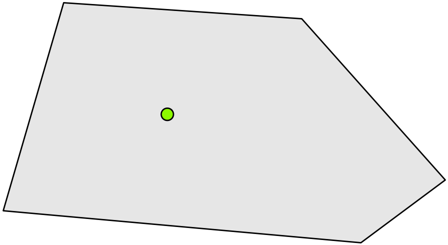

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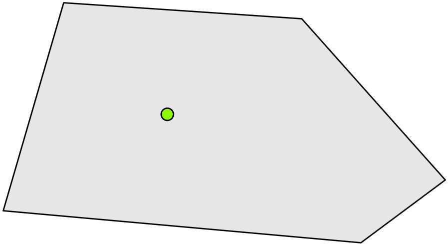

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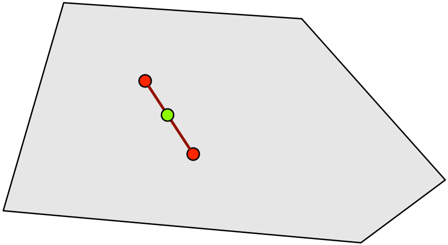

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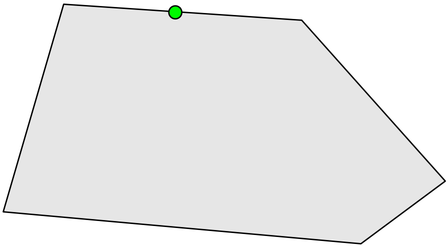

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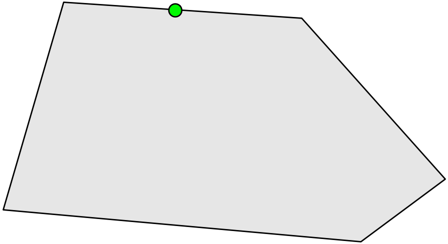

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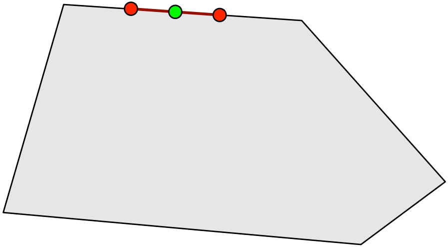

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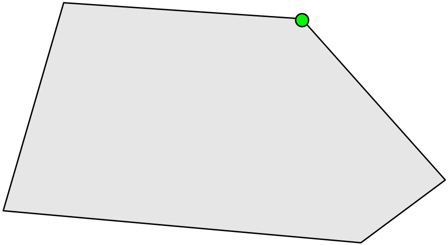

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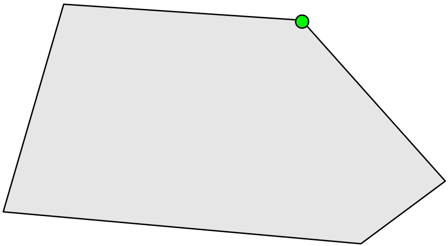

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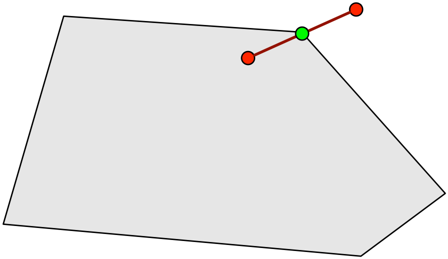

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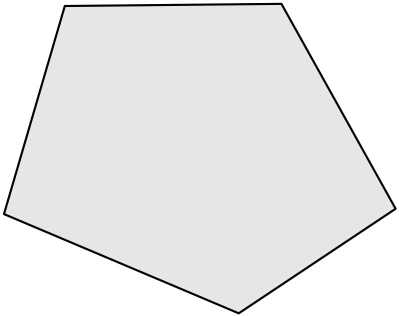
Question

What are the extreme points in the following figure?



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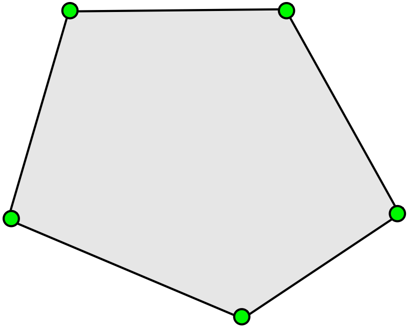

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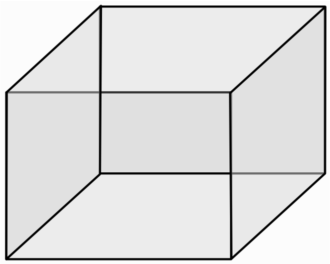

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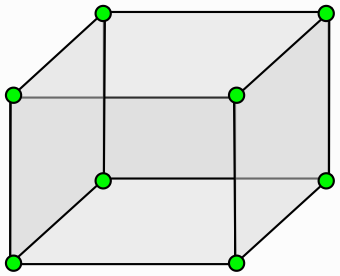

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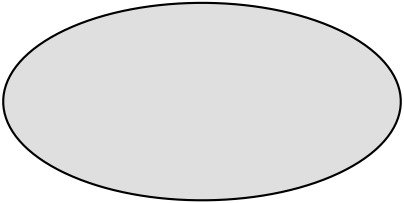

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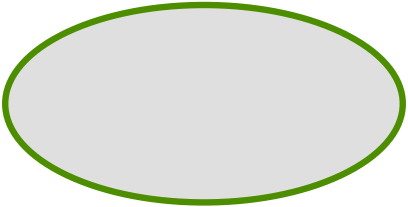

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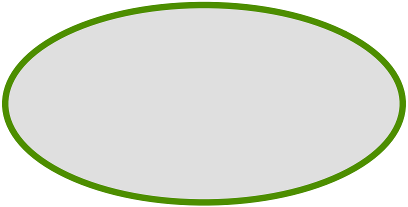

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A convex set may have an infinite number of extreme points.

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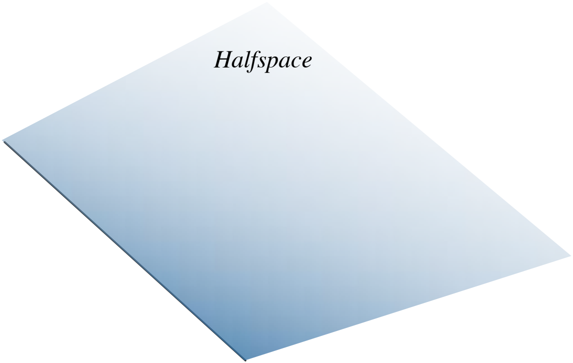

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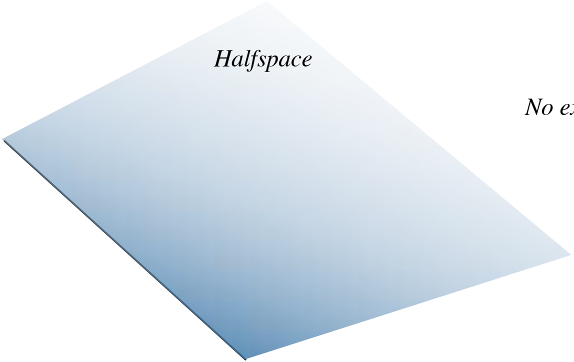

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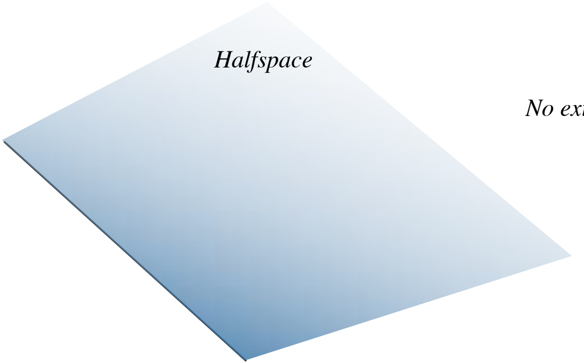

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A convex set may have NO extreme points.

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1. Characterize the extreme points in a polyhedron.

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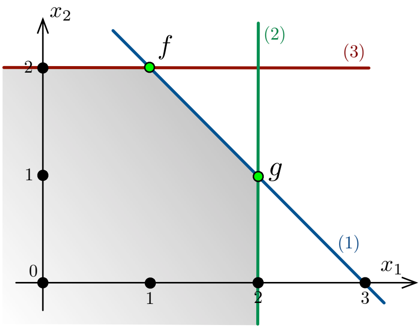
1. Characterize the extreme points in a polyhedron.
2. Characterize an extreme point for LP in Standard Equality Form.
3. Gain a geometric understanding of the Simplex algorithm.



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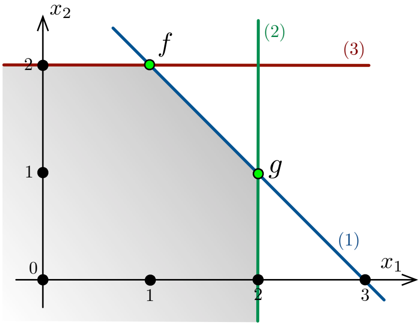

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What do the extreme points

$$f = (1,2)^\top \quad \text{and} \quad g = (2,1)^\top$$

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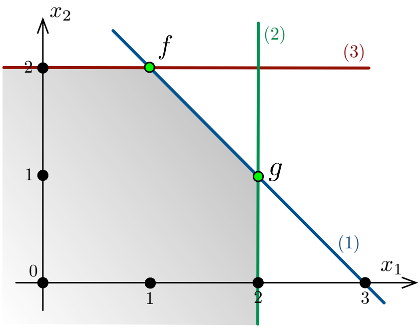

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Each satisfy $n = 2$ "independent" constraints with equality!

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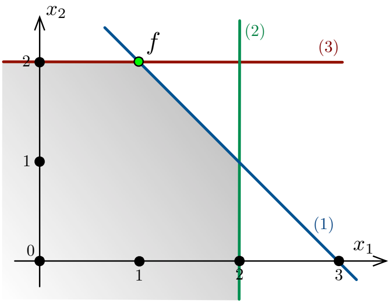

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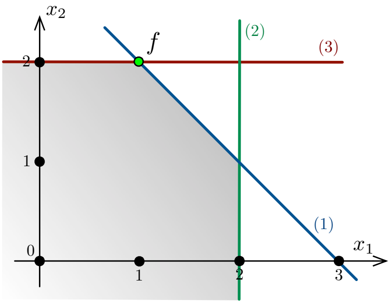

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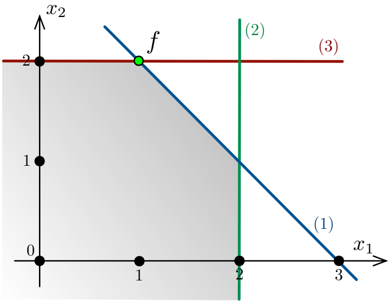

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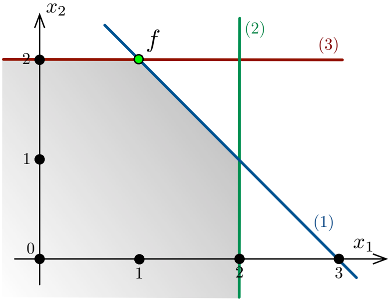

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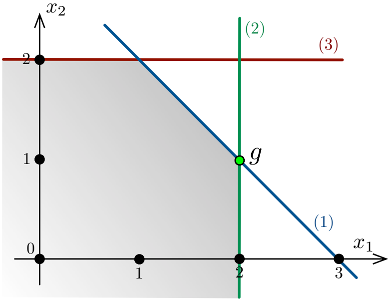

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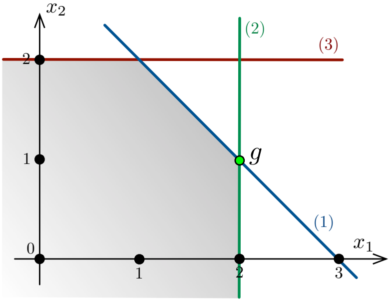

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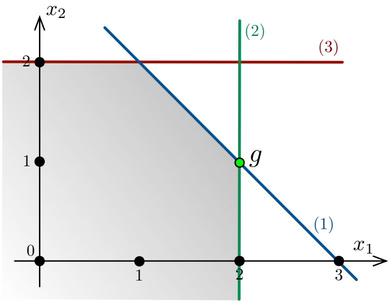

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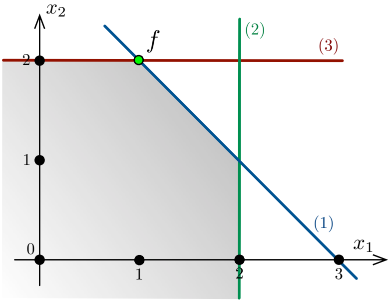

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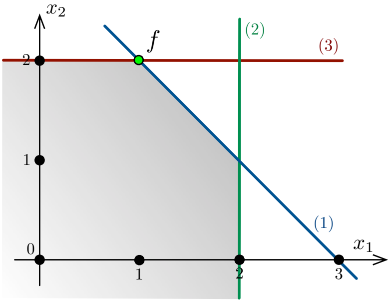

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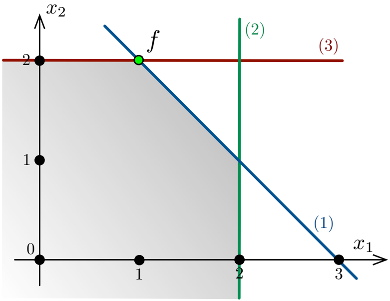

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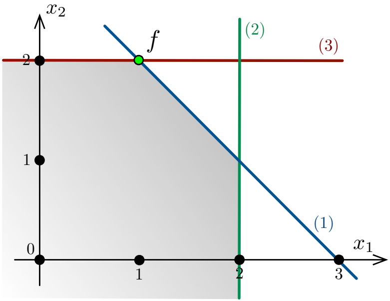

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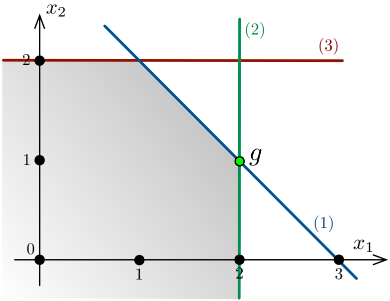

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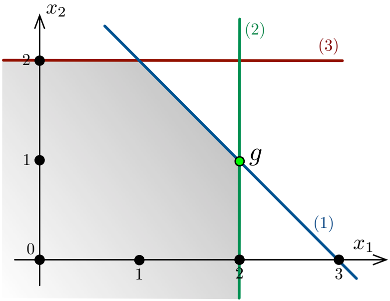

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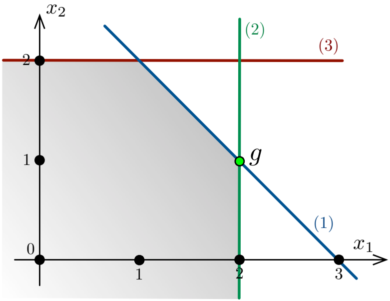

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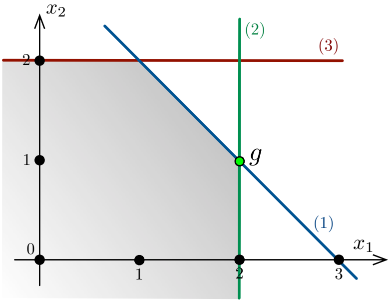

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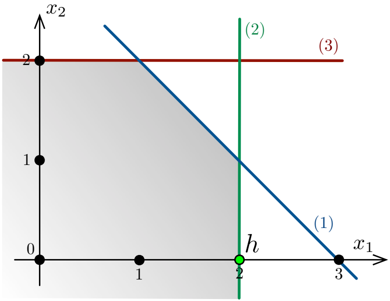

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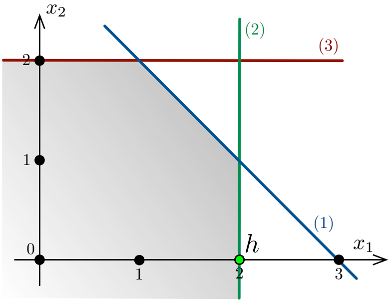

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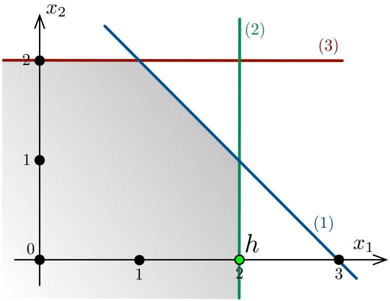

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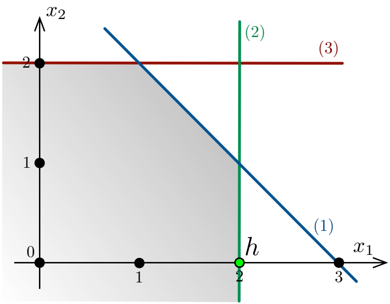

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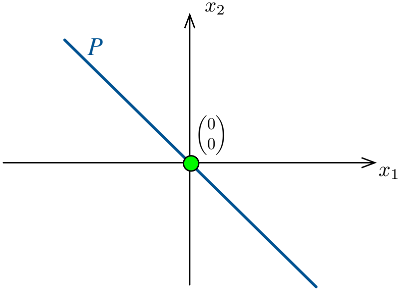

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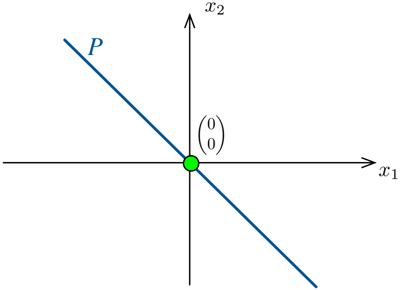

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$\bar{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has $n = 2$ rows, but $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is NOT extreme.

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Let's prove part (1).

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1. $\boxed{\text{If } rank(\bar{A}) = n\text{, then } \bar{x} \text{ is an extreme point.}}$

Proof

Suppose $\bar{x}$ is not an extreme point.

$\bar{x}$ is properly contained in a line segment with endpoints $x^{(1)}, x^{(2)} \in P$.

$\bar{x} \neq x^{(1)}, x^{(2)} \in P$ and for some λ, $0 < \lambda < 1$, $\bar{x} = \lambda x^{(1)} + (1-\lambda)x^{(2)}$.

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$\bar{A}x^{(1)} \leq \bar{b}$ and $\bar{A}x^{(2)} \leq \bar{b}$.

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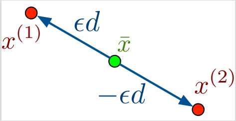

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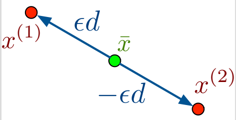

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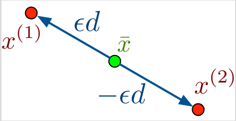

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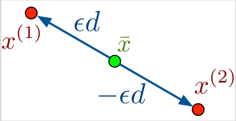

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$$A = \left(\begin{array}{rrr} 1 & 0 & -1 \\ 0 & 1 & 3 \\ \hline -1 & 0 & 1 \\ 0 & -1 & -3 \\ \hline -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right) \quad , \quad b = \left(\begin{array}{r} 2 \\ 4 \\ \hline -2 \\ -4 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$$

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$$\bar{A} = \left(\begin{array}{rrr} \color{blue}{1} & \color{blue}{0} & \color{blue}{-1} \\ \color{blue}{0} & \color{blue}{1} & \color{blue}{3} \\ \hline -1 & 0 & 1 \\ 0 & -1 & -3 \\ \hline \color{blue}{0} & \color{blue}{0} & \color{blue}{-1} \end{array}\right)$$

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Since $rank(\bar{A}) = 3$, we know that $(2,4,0)^\top$ is an extreme point!

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Since $rank(\bar{A}) = 3$, we know that $(2, 4, 0)^\top$ is an extreme point!

This is no accident...

Theorem

Let $P = \{x \geq \mathbf{0} : Ax = b\}$ where rows of A are independent. The following are equivalent:

1. $\bar{x}$ is an extreme point of P.
2. $\bar{x}$ is a basic feasible solution of P.

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Exercise

Prove the previous theorem.

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Exercise

Prove the previous theorem.

The Simplex algorithm moves from extreme points to extreme points.

Simplex - a Geometric Illustration

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$$\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}$$

Simplex - a Geometric Illustration

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}}$$

$$P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Simplex - a Geometric Illustration

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Solve using Simplex:

Simplex - a Geometric Illustration

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Solve using Simplex:

- Basis $B = \{3, 4, 5\}$, basic solution $(0, 0, 10, 6, 4)^\top$

Simplex - a Geometric Illustration

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SOLVE USING SIMPLEX:

- Basis $B = \{3, 4, 5\}$, basic solution $(0, 0, 10, 6, 4)^\top$
- Basis $B = \{1, 4, 5\}$, basic solution $(5, 0, 0, 1, 9)^\top$

Simplex - a Geometric Illustration

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Simplex - a Geometric Illustration

max	$(2,3,0,0,0)x$
s.t.	
	$x \in P_1$

$$P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

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Simplex - a Geometric Illustration

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Simplex - a Geometric Illustration

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⟹ Simplex visits extreme points of P_1 in order:

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Simplex - a Geometric Illustration

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However, we cannot draw a picture of this...

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is obtained by adding slack variables to

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

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$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}} \qquad P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

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Remark

$(0,0,10,6,4)^\top$ extreme point of P_1 $\Rightarrow$ $(0,0)^\top$ extreme point of P_2,

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

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Remark

$(0,0,10,6,4)^\top$	extreme point of P_1	$\Rightarrow$	$(0,0)^\top$	extreme point of P_2,
$(5,0,0,1,9)^\top$	extreme point of P_1	$\Rightarrow$	$(5,0)^\top$	extreme point of P_2,
$(4,2,0,0,6)^\top$	extreme point of P_1	$\Rightarrow$	$(4,2)^\top$	extreme point of P_2,
$(1,5,3,0,0)^\top$	extreme point of P_1	$\Rightarrow$	$(1,5)^\top$	extreme point of P_2.

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

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Remark

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$\Rightarrow$ Simplex visits extreme points of P_2 in order:

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

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$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}} \qquad P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Remark

$$\begin{array}{llcll}
(0,0,10,6,4)^\top & \text{extreme point of } P_1 & \Rightarrow & (0,0)^\top & \text{extreme point of } P_2, \\
(5,0,0,1,9)^\top & \text{extreme point of } P_1 & \Rightarrow & (5,0)^\top & \text{extreme point of } P_2, \\
(4,2,0,0,6)^\top & \text{extreme point of } P_1 & \Rightarrow & (4,2)^\top & \text{extreme point of } P_2, \\
(1,5,3,0,0)^\top & \text{extreme point of } P_1 & \Rightarrow & (1,5)^\top & \text{extreme point of } P_2.
\end{array}$$

➡ Simplex visits extreme points of P_2 in order:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

is obtained by adding slack variables to

$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}} \qquad P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Remark

$(0,0,10,6,4)^\top$	extreme point of P_1	$\Rightarrow$	$(0,0)^\top$	extreme point of P_2,
$(5,0,0,1,9)^\top$	extreme point of P_1	$\Rightarrow$	$(5,0)^\top$	extreme point of P_2,
$(4,2,0,0,6)^\top$	extreme point of P_1	$\Rightarrow$	$(4,2)^\top$	extreme point of P_2,
$(1,5,3,0,0)^\top$	extreme point of P_1	$\Rightarrow$	$(1,5)^\top$	extreme point of P_2.

$\Rightarrow$ Simplex visits extreme points of P_2 in order:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

is obtained by adding slack variables to

$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}} \qquad P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Remark

$$\begin{array}{llcll} (0,0,10,6,4)^\top & \text{extreme point of } P_1 & \Rightarrow & (0,0)^\top & \text{extreme point of } P_2, \\ (5,0,0,1,9)^\top & \text{extreme point of } P_1 & \Rightarrow & (5,0)^\top & \text{extreme point of } P_2, \\ (4,2,0,0,6)^\top & \text{extreme point of } P_1 & \Rightarrow & (4,2)^\top & \text{extreme point of } P_2, \\ (1,5,3,0,0)^\top & \text{extreme point of } P_1 & \Rightarrow & (1,5)^\top & \text{extreme point of } P_2. \end{array}$$

$\Rightarrow$ Simplex visits extreme points of P_2 in order:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix},$$

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

is obtained by adding slack variables to

$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}} \qquad P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Remark

$$\begin{array}{llcll} (0,0,10,6,4)^\top & \text{extreme point of } P_1 & \Rightarrow & (0,0)^\top & \text{extreme point of } P_2, \\ (5,0,0,1,9)^\top & \text{extreme point of } P_1 & \Rightarrow & (5,0)^\top & \text{extreme point of } P_2, \\ (4,2,0,0,6)^\top & \text{extreme point of } P_1 & \Rightarrow & (4,2)^\top & \text{extreme point of } P_2, \\ (1,5,3,0,0)^\top & \text{extreme point of } P_1 & \Rightarrow & (1,5)^\top & \text{extreme point of } P_2. \end{array}$$

$\Rightarrow$ Simplex visits extreme points of P_2 in order:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

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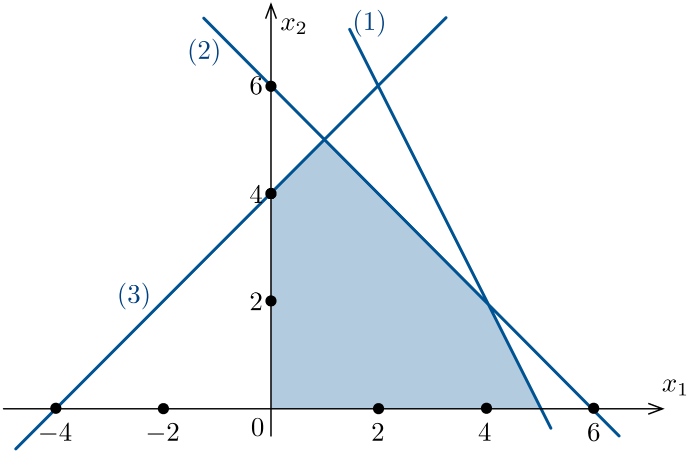

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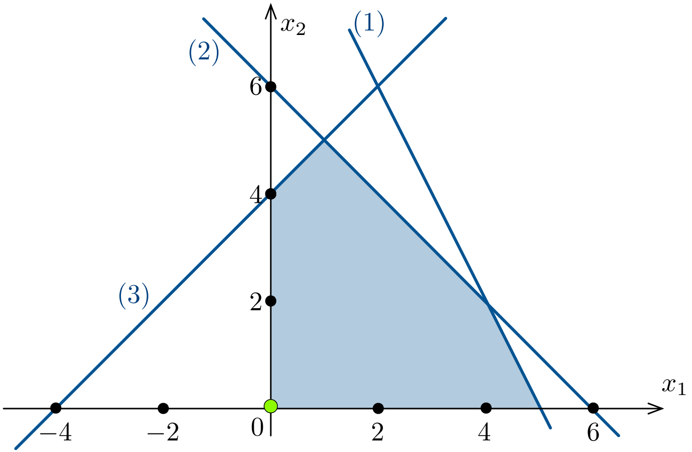

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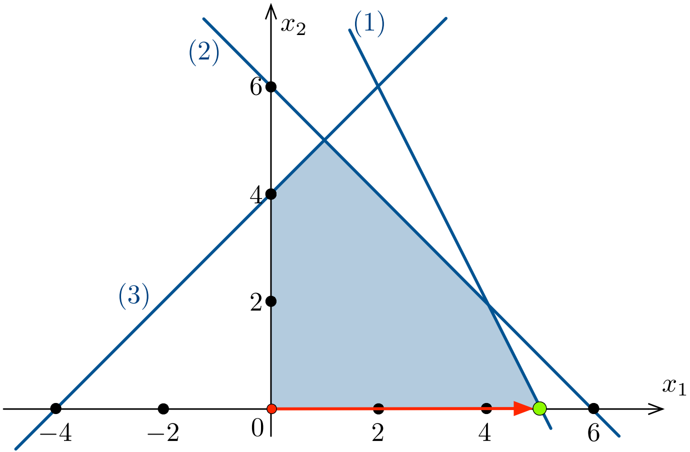

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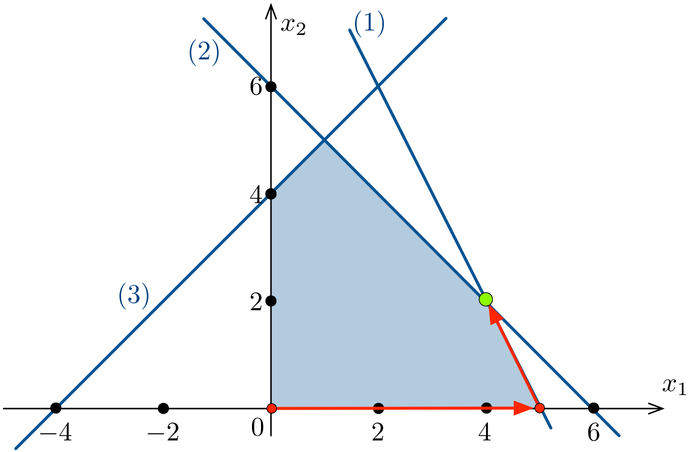

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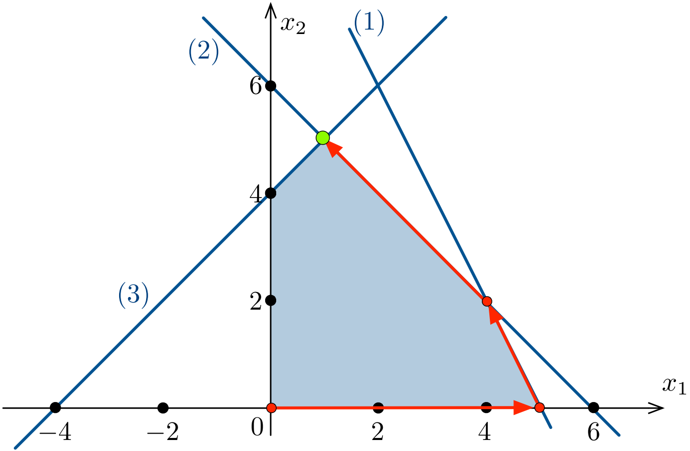

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- We showed that the Simplex algorithm moves from extreme point to extreme point.