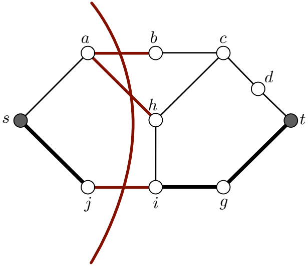
Module 4: Duality Theory (Weak Duality)

Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible $0,1$-solutions for the LP



$$\min \sum \left(c_e x_e : e \in E \right)$$

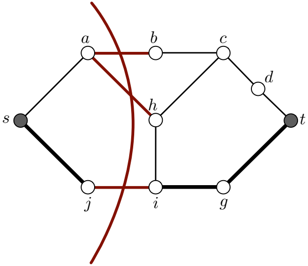
$$\text{s.t.} \sum \left(x_e \ : \ e \in \delta(U) \right) \geq 1$$

$$(U \subseteq V, s \in U, t \notin U)$$

$$x \geq \mathbb{0}$$

Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible $0, 1$-solutions for the LP



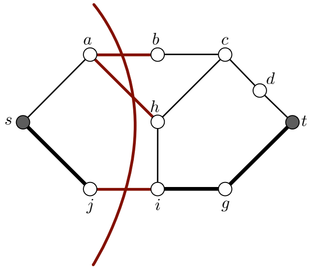
$$\min \sum \big(c_e x_e : e \in E\big)$$
$$\text{s.t.} \sum \big(x_e : e \in \delta(U)\big) \geq 1$$
$$(U \subseteq V, s \in U, t \notin U)$$
$$x \geq \mathbb{0}$$

This LP is of the form:

$$\min\{c^T x : Ax \geq b, x \geq \mathbb{0}\}$$

Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible $0, 1$-solutions for the LP



$$\begin{aligned} \min\ & \sum \big(c_e x_e : e \in E \big) \\ \text{s.t.}\ & \sum \big(x_e \ : \ e \in \delta(U) \big) \geq 1 \\ & \qquad (U \subseteq V, s \in U, t \notin U) \\ & x \geq \mathbb{0} \end{aligned}$$

This LP is of the form:

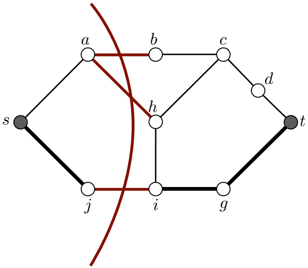
$$\min\{ c^T x \ : \ Ax \geq b, x \geq \mathbb{0} \}$$

where

- $b = \mathbb{1}$;

Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible $0, 1$-solutions for the LP



$$\begin{aligned} \min\ & \sum \bigl(c_e x_e : e \in E\bigr) \\ \text{s.t.}\ & \sum \bigl(x_e \ : \ e \in \delta(U)\bigr) \geq 1 \\ & \qquad (U \subseteq V, s \in U, t \notin U) \\ & x \geq \mathbb{0} \end{aligned}$$

This LP is of the form:

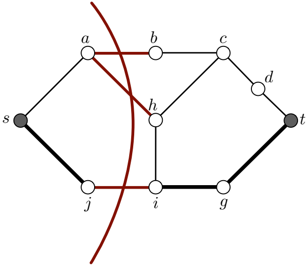
$$\min\{c^T x \ : \ Ax \geq b, x \geq \mathbb{0}\}$$

where

- $b = \mathbb{1}$;
- A has a row for every s, t-cut $\delta(U)$, and a column for every edge e; and

Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible $0, 1$-solutions for the LP



$$\begin{aligned} \min \ & \sum \big(c_e x_e : e \in E\big) \\ \text{s.t.} \ & \sum \big(x_e \ : \ e \in \delta(U)\big) \geq 1 \\ & \qquad\qquad (U \subseteq V, s \in U, t \notin U) \\ & x \geq \mathbb{0} \end{aligned}$$

This LP is of the form:

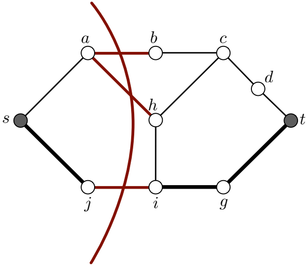
$$\min\{c^T x \ : \ Ax \geq b, x \geq \mathbb{0}\}$$

where

- $b = \mathbb{1}$;
- A has a row for every s, t-cut $\delta(U)$, and a column for every edge e; and
- $A_{Ue} = 1$ if $e \in \delta(U)$ and $A_{Ue} = 0$ otherwise.

Recap: Shortest Path Dual

$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \quad \text{(P)}$$

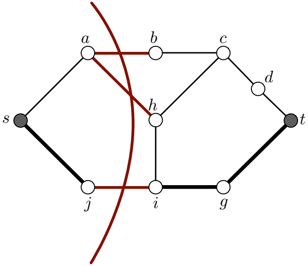


Recap: Shortest Path Dual

$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \quad \text{(P)}$$

The dual of (P) is given by

$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \quad \text{(D)}$$



Recap: Shortest Path Dual

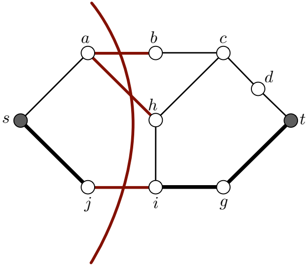
$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \quad \text{(P)}$$

The dual of (P) is given by

$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \quad \text{(D)}$$

If (P) is a shortest path LP, then we can rewrite (D) as

$$\begin{aligned} \max \; & \sum (y_U \,:\, s \in U, t \notin U) \\ \text{s.t.} \; & \sum (y_U \,:\, e \in \delta(U)) \leq c_e \\ & \qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$



Recap: Shortest Path Dual

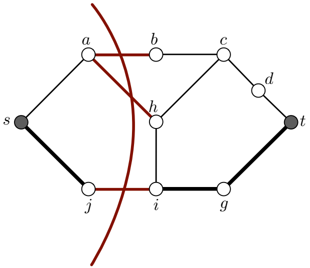
$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \quad \text{(P)}$$

The dual of (P) is given by

$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \quad \text{(D)}$$

If (P) is a shortest path LP, then we can rewrite (D) as

$$\begin{aligned} \max \ & \sum (y_U \,:\, s \in U, t \notin U) \\ \text{s.t.} \ & \sum (y_U \,:\, e \in \delta(U)) \leq c_e \\ & \qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$



Theorem

If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), then $b^T \bar{y} \leq c^T \bar{x}$.

Recap: Shortest Path Dual

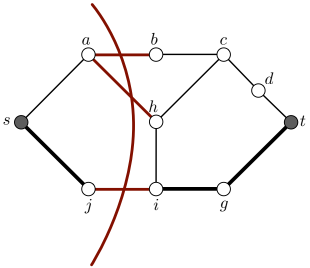
$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \quad \text{(P)}$$

The dual of (P) is given by

$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \quad \text{(D)}$$

If (P) is a shortest path LP, then we can rewrite (D) as

$$\begin{aligned} \max \ & \sum (y_U \,:\, s \in U, t \notin U) \\ \text{s.t.} \ & \sum (y_U \,:\, e \in \delta(U)) \leq c_e \\ & \qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$



Theorem

If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), then $b^T \bar{y} \leq c^T \bar{x}$.

Equivalent: y feasible widths and P an s,t-path $\longrightarrow$ $\mathbb{1}^T y \leq c(P)$

This Lecture

Question: Can we find lower-bounds on the optimal value of a general LP?

$$\max\ c^T x$$
$$\text{s.t. } Ax\ ?\ b$$
$$x\ ?\ \mathbb{0}$$

This Lecture

Question: Can we find lower-bounds on the optimal value of a general LP?

In the LP on the right,

$$Ax \; ? \; b$$

stands for a system of inequalities whose sign is one of $\leq, =$ or $\geq$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \; ? \; b \\ & x \; ? \; \mathbb{0} \end{aligned}$$

This Lecture

Question: Can we find lower-bounds on the optimal value of a general LP?

In the LP on the right,

$$Ax \; ? \; b$$

stands for a system of inequalities whose sign is one of $\leq, =$ or $\geq$, and

$$x \; ? \; \mathbb{0}$$

indicates that variables are either non-negative, non-positive, or free.

$$\begin{aligned} \max \; & c^T x \\ \text{s.t. } & Ax \; ? \; b \\ & x \; ? \; \mathbb{0} \end{aligned}$$

Recall: in the primal-dual pair

$$\min\{c^T x \, : \, Ax \geq b, x \geq \mathbb{0}\} \qquad \text{(P)}$$

$$\max\{b^T y \, : \, A^T y \leq c, y \geq \mathbb{0}\} \qquad \text{(D)}$$

This Lecture

Question: Can we find lower-bounds on the optimal value of a general LP?

In the LP on the right,

$$Ax \ ? \ b$$

stands for a system of inequalities whose sign is one of $\leq, =$ or $\geq$, and

$$x \ ? \ \mathbb{0}$$

indicates that variables are either non-negative, non-positive, or free.

$$\begin{aligned} \max \ & c^T x \\ \text{s.t. } & Ax \ ? \ b \\ & x \ ? \ \mathbb{0} \end{aligned}$$

Recall: in the primal-dual pair

$$\min\{c^T x \ : \ Ax \geq b, x \geq \mathbb{0}\} \qquad \text{(P)}$$

$$\max\{b^T y \ : \ A^T y \leq c, y \geq \mathbb{0}\} \qquad \text{(D)}$$

- each non-negative variable, x_e, in (P) corresponds to a '$\leq$'-constraint in (D), and

This Lecture

Question: Can we find lower-bounds on the optimal value of a general LP?

In the LP on the right,

$$Ax \; ? \; b$$

stands for a system of inequalities whose sign is one of $\leq, =$ or $\geq$, and

$$x \; ? \; \mathbb{0}$$

indicates that variables are either non-negative, non-positive, or free.

$$\begin{aligned} \max \; & c^T x \\ \text{s.t. } & Ax \; ? \; b \\ & x \; ? \; \mathbb{0} \end{aligned}$$

Recall: in the primal-dual pair

$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \qquad \text{(P)}$$

$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \qquad \text{(D)}$$

- each non-negative variable, x_e, in (P) corresponds to a '$\leq$'-constraint in (D), and
- each '$\geq$'-constraint in (P) corresponds to a non-negative variable y_U in (D).

Weak Duality in General

Consider the primal LP

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \ ?\ b \\ & x \ ?\ \mathbb{0} \end{aligned}$$

Weak Duality in General

Consider the primal LP

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \; ? \; b \\ & x \; ? \; \mathbb{0} \end{aligned}$$

Its dual LP is given by

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \; ? \; c \\ & y \; ? \; \mathbb{0} \end{aligned}$$

Weak Duality in General

Consider the primal LP

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \; ? \; b \\ & x \; ? \; \mathbb{0} \end{aligned}$$

Its dual LP is given by

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \; ? \; c \\ & y \; ? \; \mathbb{0} \end{aligned}$$

Question: What are the question marks?

Weak Duality in General

Consider the primal LP

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \ ?\ b \\ & x \ ?\ \mathbb{0} \end{aligned}$$

Its dual LP is given by

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \ ?\ c \\ & y \ ?\ \mathbb{0} \end{aligned}$$

Question: What are the question marks?

A: As before:

primal variables $\equiv$ dual constraints
primal constraints $\equiv$ dual variables

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$		
		$\leq$ constraint	≥ 0 variable		
max	$c^\top x$	$=$ constraint	free variable	min	$b^\top y$
subject to		$\geq$ constraint	≤ 0 variable	subject to	
	Ax **?** b	≥ 0 variable	$\geq$ constraint		$A^\top y$ **?** c
	x **?** $\mathbb{0}$	free variable	$=$ constraint		y **?** $\mathbb{0}$
		≤ 0 variable	$\leq$ constraint		

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax\ ?\ b$	≥ 0 variable	$\geq$ constraint	$A^\top y\ ?\ c$
$x\ ?\ \mathbb{0}$	free variable	$=$ constraint	$y\ ?\ \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 1:

$$\max\ (1,0,2)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \geq 0, x_3 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
max $\quad c^\top x$	$\le$ constraint	≥ 0 variable	min $\quad b^\top y$
subject to	$=$ constraint	free variable	subject to
$Ax \; ? \; b$	$\ge$ constraint	≤ 0 variable	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	≥ 0 variable	$\ge$ constraint	$y \; ? \; \mathbb{0}$
	free variable	$=$ constraint	
	≤ 0 variable	$\le$ constraint	

Example 1:

$$\max \; (1,0,2)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \le \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \ge 0, x_3 \text{ free}$$

Its dual LP:

$$\min \; (3,4)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \; ? \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$y \; ? \; \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, 0$	free variable	$=$ constraint	$y \,?\, 0$
	≤ 0 variable	$\leq$ constraint	

Example 1:

$$\max\ (1,0,2)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \geq 0, x_3 \text{ free}$$

Its dual LP:

$$\min\ (3,4)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \ ? \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
Ax ? b	≥ 0 variable	$\geq$ constraint	$A^\top y$? c
x ? $\mathbb{0}$	free variable	$=$ constraint	y ? $\mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 1:

$$\max\ (1,0,2)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \geq 0, x_3 \text{ free}$$

Its dual LP:

$$\min\ (3,4)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \begin{matrix} \geq \\ \geq \\ = \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(P_{\max})$			$(P_{\min})$
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
Ax ? b	≥ 0 variable	$\geq$ constraint	$A^\top y$? c
x ? $\mathbb{0}$	free variable	$=$ constraint	y ? $\mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\min\ d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

(P_{max})			(P_{min})		
max $c^\top x$ subject to $Ax \,?\, b$, $x \,?\, \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \,?\, c$, $y \,?\, \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\min\ d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

To compute dual LP, check right-hand side of table:

$$\max\ e^T x \qquad \text{(D)}$$
$$\text{s.t. } Wx \ ?\ d$$
$$x \ ?\ \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, \mathbb{0}$	free variable	$=$ constraint	$y \,?\, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\min\ d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

To compute dual LP, check right-hand side of table:

$$\max\ e^T x \qquad \text{(D)}$$
$$\text{s.t. } Wx \ ?\ d$$
$$x \geq \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \, ? \, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \, ? \, c$
$x \, ? \, \mathbb{0}$	free variable	$=$ constraint	$y \, ? \, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\min\ d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

To compute dual LP, check right-hand side of table:

$$\max\ e^T x \qquad \text{(D)}$$
$$\text{s.t. } Wx \leq d$$
$$x \geq \mathbb{0}$$

Primal-Dual Pairs

Example 2:

$$\begin{aligned} \min\ & d^T y \\ \text{s.t. } & W^T y \geq e \\ & y \geq \mathbb{0} \end{aligned} \tag{P}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\begin{aligned} \max\ & e^T x \\ \text{s.t. } & W x \leq d \\ & x \geq \mathbb{0} \end{aligned} \tag{D}$$

Primal-Dual Pairs

Example 2:

$$\min\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \geq b$$
$$x \geq \mathbb{0}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\max\ b^T x \qquad \text{(D)}$$
$$\text{s.t. } A^T y \leq c$$
$$y \geq \mathbb{0}$$

Primal-Dual Pairs

Example 2:

$$\min \ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \geq b$$
$$x \geq \mathbb{0}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\max \ b^T x \qquad \text{(D)}$$
$$\text{s.t. } A^T y \leq c$$
$$y \geq \mathbb{0}$$

This is consistent with the earlier discussion we had!

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, \mathbb{0}$	free variable	$=$ constraint	$y \,?\, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
max $c^\top x$ subject to $Ax \;?\; b$ $x \;?\; \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \;?\; c$ $y \;?\; \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \;?\; \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y \;?\; \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$		
max	$c^\top x$	$\le$ constraint	≥ 0 variable	min	$b^\top y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \,?\, b$	$\ge$ constraint	≤ 0 variable		$A^\top y \,?\, c$
	$x \,?\, \mathbb{0}$	≥ 0 variable	$\ge$ constraint		$y \,?\, \mathbb{0}$
		free variable	$=$ constraint		
		≤ 0 variable	$\le$ constraint		

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \ge \\ \le \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \ge 0, x_2 \text{ free}, x_3 \ge 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \ ?\ \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \le 0, y_2 \ge 0, y_3 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			
max $c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} 0$	$\leq$ constraint $=$ constraint $\geq$ constraint ≥ 0 variable free variable ≤ 0 variable	≥ 0 variable free variable ≤ 0 variable $\geq$ constraint $=$ constraint $\leq$ constraint	min $b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} 0$

The header $(\mathrm{P}_{\max})$ spans the first two columns and $(\mathrm{P}_{\min})$ spans the last two.

Example 3:

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
max $c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} \mathbb{0}$	$\le$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} \mathbb{0}$
	$=$ constraint	free variable	
	$\ge$ constraint	≤ 0 variable	
	≥ 0 variable	$\ge$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\le$ constraint	

Theorem

Let $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$ represent the above.

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

(P_{max})			(P_{min})
max $c^\top x$ subject to $Ax\ ?\ b$ $x\ ?\ \mathbb{0}$	$\leq$ constraint $=$ constraint $\geq$ constraint ≥ 0 variable free variable ≤ 0 variable	≥ 0 variable free variable ≤ 0 variable $\geq$ constraint $=$ constraint $\leq$ constraint	min $b^\top y$ subject to $A^\top y\ ?\ c$ $y\ ?\ \mathbb{0}$

Theorem

Let (P_{max}) and (P_{min}) represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

Primal-Dual Pairs

Theorem

Let ($\mathsf{P}_{\mathsf{max}}$) and ($\mathsf{P}_{\mathsf{min}}$) represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \le b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for ($\mathsf{P}_{\mathsf{max}}$), and $\bar{y}$ is optimal for ($\mathsf{P}_{\mathsf{min}}$).

Example 3 (continued):

$$\begin{aligned} &\max\ (12, 26, 20)x \qquad\qquad (\mathsf{P}) \\ &\text{s.t.}\ \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \ge \\ \le \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix} \\ &\quad x_1 \ge 0, x_2 \text{ free}, x_3 \ge 0 \end{aligned}$$

Its dual LP:

$$\begin{aligned} &\min\ (-2, 2, 13)y \qquad\qquad (\mathsf{D}) \\ &\text{s.t.}\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \ge \\ = \\ \ge \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix} \\ &\quad y_1 \le 0, y_2 \ge 0, y_3 \text{ free} \end{aligned}$$

Feasible solutions: $\bar{x} = (5, -3, 0)^T$ and $\bar{y} = (0, 4, -2)^T$.

Primal-Dual Pairs

Theorem

Let (P_{max}) and (P_{min}) represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

Example 3 (continued):

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \text{ free}$$

Feasible solutions: $\bar{x} = (5, -3, 0)^T$ and $\bar{y} = (0, 4, -2)^T$.
Since $(12, 26, 20)\bar{x} = (-2, 2, 13)\bar{y} = -18 \longrightarrow$ both are optimal!

Proving the General Weak Duality Theorem

(P_{max})				(P_{min})	
max	$c^\top x$	$\leq$ constraint	≥ 0 variable	min	$b^\top y$
subject to		$=$ constraint	free variable	subject to	
	$Ax \mathbin{?} b$	$\geq$ constraint	≤ 0 variable		$A^\top y \mathbin{?} c$
	$x \mathbin{?} \mathbb{0}$	≥ 0 variable	$\geq$ constraint		$y \mathbin{?} \mathbb{0}$
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & \text{row}_i(A)x \leq b_i \ (i \in R_1) \\
& \text{row}_i(A)x \geq b_i \ (i \in R_2) \\
& \text{row}_i(A)x = b_i \ (i \in R_3) \\
& x_j \geq 0 \ (j \in C_1) \\
& x_j \leq 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Proving the General Weak Duality Theorem

(P max)			(P min)	
max $c^\top x$ subject to $Ax \mathbin{?} b$, $x \mathbin{?} \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \mathbin{?} c$, $y \mathbin{?} \mathbb{0}$	
	$=$ constraint	free variable		
	$\geq$ constraint	≤ 0 variable		
	≥ 0 variable	$\geq$ constraint		
	free variable	$=$ constraint		
	≤ 0 variable	$\leq$ constraint		

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & \mathrm{row}_i(A)x \leq b_i \ \ (i \in R_1) \\
& \mathrm{row}_i(A)x \geq b_i \ \ (i \in R_2) \\
& \mathrm{row}_i(A)x = b_i \ \ (i \in R_3) \\
& x_j \geq 0 \ \ (j \in C_1) \\
& x_j \leq 0 \ \ (j \in C_2) \\
& x_j \text{ free } \ (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & \mathrm{col}_j(A)^T y \geq c_j \ \ (j \in C_1) \\
& \mathrm{col}_j(A)^T y \leq c_j \ \ (j \in C_2) \\
& \mathrm{col}_j(A)^T y = c_j \ \ (j \in C_3) \\
& y_i \geq 0 \ \ (i \in R_1) \\
& y_i \leq 0 \ \ (i \in R_2) \\
& y_i \text{ free } \ (i \in R_3)
\end{aligned}
$$

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned}
&\max\ c^T x \\
&\quad \text{s.t. } \mathsf{row}_i(A)x \le b_i \ (i \in R_1) \\
&\qquad\quad \mathsf{row}_i(A)x \ge b_i \ (i \in R_2) \\
&\qquad\quad \mathsf{row}_i(A)x = b_i \ (i \in R_3) \\
&\qquad\quad x_j \ge 0 \ (j \in C_1) \\
&\qquad\quad x_j \le 0 \ (j \in C_2) \\
&\qquad\quad x_j \text{ free } (j \in C_3)
\end{aligned}$$

Its dual according to the table:

$$\begin{aligned}
&\min\ b^T y \\
&\quad \text{s.t. } \mathsf{col}_j(A)^T y \ge c_j \ (j \in C_1) \\
&\qquad\quad \mathsf{col}_j(A)^T y \le c_j \ (j \in C_2) \\
&\qquad\quad \mathsf{col}_j(A)^T y = c_j \ (j \in C_3) \\
&\qquad\quad y_i \ge 0 \ (i \in R_1) \\
&\qquad\quad y_i \le 0 \ (i \in R_2) \\
&\qquad\quad y_i \text{ free } (i \in R_3)
\end{aligned}$$

We can rewrite the above LPs using slack variables!

Proving the General Weak Duality Theorem

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & Ax + s = b \\
& s_i \geq 0 \ (i \in R_1) \\
& s_i \leq 0 \ (i \in R_2) \\
& s_i = 0 \ (i \in R_3) \\
& x_j \geq 0 \ (j \in C_1) \\
& x_j \leq 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & A^T y + w = c \qquad (\star) \\
& w_j \leq 0 \ (j \in C_1) \\
& w_j \geq 0 \ (j \in C_2) \\
& w_j = 0 \ (j \in C_3) \\
& y_i \geq 0 \ (i \in R_1) \\
& y_i \leq 0 \ (i \in R_2) \\
& y_i \text{ free } (i \in R_3)
\end{aligned}
$$

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \qquad (\star) \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned}$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned} \qquad (\star)$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$.

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} &\max\ c^T x \\ &\text{s.t. } Ax + s = b \\ &\quad s_i \geq 0\ (i \in R_1) \\ &\quad s_i \leq 0\ (i \in R_2) \\ &\quad s_i = 0\ (i \in R_3) \\ &\quad x_j \geq 0\ (j \in C_1) \\ &\quad x_j \leq 0\ (j \in C_2) \\ &\quad x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} &\min\ b^T y \\ &\text{s.t. } A^T y + w = c \qquad (\star) \\ &\quad w_j \leq 0\ (j \in C_1) \\ &\quad w_j \geq 0\ (j \in C_2) \\ &\quad w_j = 0\ (j \in C_3) \\ &\quad y_i \geq 0\ (i \in R_1) \\ &\quad y_i \leq 0\ (i \in R_2) \\ &\quad y_i \text{ free } (i \in R_3) \end{aligned}$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s}$$

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \qquad (\star) \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned}$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} \stackrel{(\star)}{=}$$
$$(c - \bar{w})^T\bar{x} + \bar{y}^T\bar{s} = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}.$$

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \qquad (\star) \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned}$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} \overset{(\star)}{=} (c - \bar{w})^T\bar{x} + \bar{y}^T\bar{s} = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}.$$

We can show that $\bar{w}^T\bar{x} \leq 0$ and $\bar{y}^T\bar{s} \geq 0$

Proving the General Weak Duality Theorem

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & Ax + s = b \\
& s_i \geq 0 \ (i \in R_1) \\
& s_i \leq 0 \ (i \in R_2) \\
& s_i = 0 \ (i \in R_3) \\
& x_j \geq 0 \ (j \in C_1) \\
& x_j \leq 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & A^T y + w = c \qquad (\star) \\
& w_j \leq 0 \ (j \in C_1) \\
& w_j \geq 0 \ (j \in C_2) \\
& w_j = 0 \ (j \in C_3) \\
& y_i \geq 0 \ (i \in R_1) \\
& y_i \leq 0 \ (i \in R_2) \\
& y_i \text{ free } (i \in R_3)
\end{aligned}
$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$
\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} \overset{(\star)}{=} (c - \bar{w})^T\bar{x} + \bar{y}^T\bar{s} = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}.
$$

We can show that $\bar{w}^T\bar{x} \leq 0$ and $\bar{y}^T\bar{s} \geq 0 \longrightarrow \quad \bar{y}^T b \geq c^T\bar{x}$

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

Consequences of Weak Duality

Theorem

Let ($\mathrm{P}_{\max}$) and ($\mathrm{P}_{\min}$) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for ($\mathrm{P}_{\max}$), and $\bar{y}$ is optimal for ($\mathrm{P}_{\min}$).

(i) ($\mathrm{P}_{\max}$) is unbounded $\longrightarrow$ ($\mathrm{P}_{\min}$) infeasible

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}).

Consequences of Weak Duality

Theorem

Let ($P_{\max}$) and ($P_{\min}$) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \le b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for ($P_{\max}$), and $\bar{y}$ is optimal for ($P_{\min}$).

(i) ($P_{\max}$) is unbounded $\longrightarrow$ ($P_{\min}$) infeasible

(ii) ($P_{\min}$) is unbounded $\longrightarrow$ ($P_{\max}$) infeasible

(iii) ($P_{\max}$) and ($P_{\min}$) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for ($P_{\min}$). By weak duality $\longrightarrow$ $c^T\bar{x} \le b^T\bar{y}$ for all $\bar{x}$ feasible for ($P_{\max}$), and hence the latter is bounded.

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T \bar{x} \leq b^T \bar{y}$$

If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}). By weak duality $\longrightarrow$ $c^T \bar{x} \leq b^T \bar{y}$ for all $\bar{x}$ feasible for (P_{max}), and hence the latter is bounded.

(ii) Similar to (i)

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T \bar{x} \leq b^T \bar{y}$$

If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}). By weak duality $\longrightarrow$ $c^T \bar{x} \leq b^T \bar{y}$ for all $\bar{x}$ feasible for (P_{max}), and hence the latter is bounded.

(ii) Similar to (i)

(iii) weak duality $\longrightarrow$ both (P_{max}) and (P_{min}) bounded

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}). By weak duality $\longrightarrow$ $c^T\bar{x} \leq b^T\bar{y}$ for all $\bar{x}$ feasible for (P_{max}), and hence the latter is bounded.

(ii) Similar to (i)

(iii) weak duality $\longrightarrow$ both (P_{max}) and (P_{min}) bounded

Fundamental Theorem of LP $\longrightarrow$ Both LPs must have an optimal solution! $\square$

(P_{max})		(P_{min})	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, \mathbb{0}$	free variable	$=$ constraint	$y \,?\, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Recap

- We can use the above table to compute duals of general LPs

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, 0$	free variable	$=$ constraint	$y \,?\, 0$
	≤ 0 variable	$\leq$ constraint	

Recap

- We can use the above table to compute duals of general LPs
- Weak duality theorem: if $\bar{x}$ and $\bar{y}$ are feasible for $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$, then:

$$c^T \bar{x} \leq b^T \bar{y}$$

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \,?\, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \,?\, c$
$x \,?\, \mathbb{0}$	free variable	$=$ constraint	$y \,?\, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Recap

- We can use the above table to compute duals of general LPs
- Weak duality theorem: if $\bar{x}$ and $\bar{y}$ are feasible for $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$, then:

$$c^T \bar{x} \leq b^T \bar{y}$$

  Both are optimal if equality holds!