Medical Imaging

Prof. Dr. Tobias Knopp

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Institute für Biomedizinische Bildgebung

Signal Processing

Signal Processing

- Understanding tomographic imaging techniques requires profound knowledge of signal processing
- In particular Fourier analysis plays an important role
- Basics of signal processing will be recapitulated

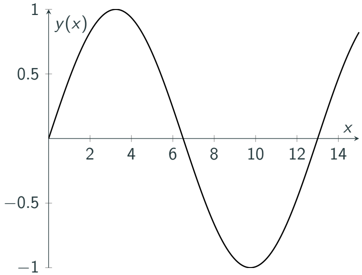
Signal

- A **signal** can be seen as a message that is send from a *sender* to a *recipient*.
- Usually they transport a physical quantity
- Signals can either be discrete or continuous (i.e. a function of $\mathbb{N}$ or $\mathbb{R}$)
- Typically a signal depends on *time* and/or *space*

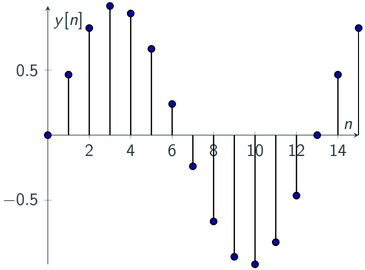
Signal Examples

- $s(x)$: spatial 1D signal, e.g. detector array in CT
- $f(x, y)$: spatial 2D signal, e.g. slice through an object in CT
- $c(t)$: temporal 1D signal, e.g. ECG

continuous signal



discrete signal



Fundamental Signals

- Heaviside step function

$$\operatorname{step}(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases} \tag{1}$$

- Rectangular function

$$\operatorname{rect}(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2} \\ 0.5, & \text{for } |x| = \frac{1}{2} \\ 0, & \text{for } |x| > \frac{1}{2} \end{cases} \tag{2}$$

- Sinc function

$$\operatorname{si}(x) = \operatorname{sinc}(x) = \frac{sin(\pi x)}{\pi x} \tag{3}$$

- Dirac delta distribution

$$\delta(x - x_0) = \begin{cases} 0, & \text{for } x \neq x_0 \\ \infty, & \text{for } x = x_0 \end{cases}$$
$$= \lim_{\tilde{x} \to 0} \frac{1}{\tilde{x}} \mathrm{rect}(\frac{x - x_0}{\tilde{x}})$$

Fundamental Signals

- The Dirac delta distribution is no function in a classical sense as ∞ can only be considered in the limes and is not a valid function value.
- The Dirac delta distribution can be defined over the integral

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \, \mathrm{d}x = f(x_0) \tag{4}$$

In particular it holds that

$$\int_{-\infty}^{\infty} \delta(x - x_0) \, \mathrm{d}x = 1 \tag{5}$$

I.e. Dirac delta distribution has unit area.

Systems

- A system $\mathcal{L}$ takes as input a function $f(x)$ and outputs a function $g(x)$

$$g(x) = \mathcal{L}\{f(x)\}$$

- A system is linear if

$$\mathcal{L}\{\sum_i a_i f_i(x)\} = \sum_i a_i \mathcal{L}\{f_i(x)\}$$

- A system is said to be *time invariant* / *shift invariant* if

$$\mathcal{L}\{f(x - x_0)\} = \mathcal{L}\{f\}(x - x_0).$$

  This means a shift of the input signal by x_0 leads to a shift by x_0 of the output signal.

- A system is said to be LTI or LSI if it is linear and time / shift invariant.

Impulse Response and Convolution

Any LSI system can be described by a convolution integral

$$\begin{aligned} g(x) &= \mathcal{L}\{f(x)\} \\ &= \int_{-\infty}^{\infty} f(\tilde{x})h(x-\tilde{x})\,\mathrm{d}\tilde{x} \\ &= (f * h)(x) \end{aligned}$$

h is the so-called *impulse response* or *point-spread function (PSF)* that is obtained by applying a Dirac delta to the LSI system

$$\begin{aligned} \mathcal{L}\{\delta(x)\} &= \int_{-\infty}^{\infty} \delta(\tilde{x})h(x-\tilde{x})\,\mathrm{d}\tilde{x} \\ &\underset{\text{subst. } y=x-\tilde{x}}{=} \int_{-\infty}^{\infty} \delta(x-y)h(y)\,\mathrm{d}y \\ &\underset{\delta \text{ is even}}{=} \int_{-\infty}^{\infty} \delta(y-x)h(y)\,\mathrm{d}y = h(x) \end{aligned}$$

Derivation of the Convolution Integral

Idea: express $f(x)$ as a sum of shifted rectangular functions

$$f(x) = \lim_{x_0 \to 0} \sum_{n=-\infty}^{\infty} f(nX_0)\mathrm{rect}\left(\frac{x - nX_0}{X_0}\right)$$

Applying LSI properties yields

$$\begin{aligned} g(x) = \mathcal{L}\{f(x)\} &= \lim_{X_0 \to 0} \sum_{n=-\infty}^{\infty} f(nX_0)\mathcal{L}\left\{\mathrm{rect}\left(\frac{x - nX_0}{X_0}\right)\right\} \\ &= \int_{-\infty}^{\infty} f(\tilde{x})\mathcal{L}\{\delta(x - \tilde{x})\}\,\mathrm{d}\tilde{x} \\ &= \int_{-\infty}^{\infty} f(\tilde{x})h(x - \tilde{x})\,\mathrm{d}\tilde{x} \end{aligned}$$

Fourier Transformation

It is known from analysis lectures that (almost) any p-periodic function $s(x)$ can be expanded into a *Fourier series*

$$s(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{2\pi\mathrm{i}\frac{nx}{p}}$$

consisting of complex sinus functions $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$. The Fourier coefficients c_n can be calculated by

$$c_n = \frac{1}{p}\int_{-p/2}^{p/2} s(x)\mathrm{e}^{-2\pi\mathrm{i}\frac{nx}{p}}\,\mathrm{d}x$$

From Fourier Series to Fourier Transformation

In order to also express non-periodic functions by Fourier series (i.e. any general signal/function) one can consider the limes $p \to \infty$

$$s(x) = \lim_{p \to \infty} \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{2\pi \mathrm{i} \frac{nx}{p}}$$
$$\underset{f := \frac{n}{p}}{=} \int_{-\infty}^{\infty} S(f) \mathrm{e}^{2\pi \mathrm{i} f x} \, \mathrm{d}f =: \mathcal{F}^{-1}\{S(f)\}$$

Here, $f := \frac{n}{p}$ is the frequency and $S(f) = S(\frac{n}{p}) = c_n$ is a continuous function of frequency. This is in contrast to the discrete spectrum of the periodic Fourier series.

From Fourier Series to Fourier Transformation

The Fourier transform $S(f)$ of $s(x)$ can be calculated by

$$S(f) = \int_{-\infty}^{\infty} s(x)\mathrm{e}^{-2\pi \mathrm{i} f x}\,\mathrm{d}x =: \mathcal{F}\{s(x)\}$$

$S(f)$ is often called the *spectrum* of $s(x)$. The Fourier relation is often indicated by

$$s(x) \;\frac{\quad}{}\!\circ\, S(f)$$

Example

The Fourier transformation of the rectangular function can be calculated to be

$$\begin{aligned}
\mathcal{F}\{\mathrm{rect}(x)\} &= \int_{-\infty}^{\infty} \mathrm{rect}(x)\mathrm{e}^{-2\pi \mathrm{i} f x}\,\mathrm{d}x \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i} f x}\,\mathrm{d}x \\
&= \frac{\mathrm{i}}{2\pi f}\left(\mathrm{e}^{-\mathrm{i}\pi f} - \mathrm{e}^{\mathrm{i}\pi f}\right) \\
&= \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(f)
\end{aligned}$$

In the last step the euler formula $\sin(f) = \frac{\mathrm{e}^{\mathrm{i}f} - \mathrm{e}^{-\mathrm{i}f}}{2\mathrm{i}}$ has been used.

Transfer Function

Instead of describing an LSI system as a convolution with the PSF $h(x)$ one can equivalently describe it in Fourier space by considering the *transfer function* $H(f) = \mathcal{F}\{h(x)\}$.

Theorem
A convolution $g(x) = (s * h)(x)$ in spatial space corresponds to a multiplication in Fourier space:

$$G(f) = S(f)H(f)$$

where $G(f) = \mathcal{F}\{g(x)\}$, $S(f) = \mathcal{F}\{s(x)\}$, and $H(f) = \mathcal{F}\{h(x)\}$.

Proof

$$
\begin{aligned}
G(f) = \mathcal{F}\{g(x)\} &= \mathcal{F}\{(s * h)(x)\} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\tilde{x}) h(x - \tilde{x}) \, \mathrm{d}\tilde{x} \mathrm{e}^{-2\pi \mathrm{i} f x} \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\tilde{x}) h(x - \tilde{x}) \mathrm{e}^{-2\pi \mathrm{i} f (x - \tilde{x})} \mathrm{e}^{-2\pi \mathrm{i} f \tilde{x}} \, \mathrm{d}\tilde{x} \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\tilde{x}) h(z) \mathrm{e}^{-2\pi \mathrm{i} f \tilde{x}} \mathrm{e}^{-2\pi \mathrm{i} f z} \, \mathrm{d}\tilde{x} \, \mathrm{d}z \\
&= \int_{-\infty}^{\infty} s(\tilde{x}) \mathrm{e}^{-2\pi \mathrm{i} f \tilde{x}} \, \mathrm{d}\tilde{x} \int_{-\infty}^{\infty} h(z) \mathrm{e}^{-2\pi \mathrm{i} f z} \, \mathrm{d}z \\
&= S(f) H(f)
\end{aligned}
$$

Application of the Convolution theorem

- Efficient application of filter (low pass, high pass)
- Deconvolution / image sharpening

Discrete Fourier transformation

In order to numerically calculate Fourier coefficients one has to sample the Fourier integral a discrete and equidistant sampling points.

This makes the Fourier coefficients periodic so that the data in both domains (spatial and frequency) is discrete (line spectrum) and periodic.

Given a sequence $s_0, \ldots, s_{N-1}$ the discrete Fourier transformation (DFT) is defines as

$$S_m = \sum_{n=0}^{N-1} s_n \mathrm{e}^{-2\pi \mathrm{i} \frac{nm}{N}}, \quad m = 0, \ldots, N-1$$

Discrete Fourier transformation

The DFT is a unitary transformation and its inverse can be calculated by

$$s_n = \frac{1}{N}\sum_{m=0}^{N-1} S_m \mathrm{e}^{2\pi\mathrm{i}\frac{nm}{N}}, \quad n = 0, \ldots, N-1$$

When defining the vectors $\boldsymbol{S} := (S_m)_{m=0}^{N-1}$ and $\boldsymbol{s} := (s_n)_{n=0}^{N-1}$, and the discrete Fourier matrix $\boldsymbol{F} := \left(\mathrm{e}^{-2\pi\mathrm{i}\frac{nm}{N}}\right)_{m=0,\ldots,N-1;n=0,\ldots,N-1}$ the DFT can be written in matrix-vector form

$$\boldsymbol{S} = \boldsymbol{F}\boldsymbol{s}$$

Fast Fourier Transformation

A naive implementation of the DFT would require $\mathcal{O}(N^2)$ arithmetic operations.

The fast Fourier transformation is a fast algorithm capable of carrying out FFT in only $\mathcal{O}(N \log N)$. It uses a recursive divide and conquer principle.

Important is that the FFT can only be applied to equidistant sampling positions.

Fast Fourier Transformation

Assumption: We will derive the DFT for $N = 2^r$.

Basic idea: split the sum into two sums, which are itself regular DFTs.

$$
\begin{aligned}
S_m &= \sum_{n=0}^{N-1} s_n \mathrm{e}^{-2\pi\mathrm{i}\frac{nm}{N}} \\
&= \sum_{n=0}^{\frac{N}{2}-1} s_n \mathrm{e}^{-2\pi\mathrm{i}\frac{nm}{N}} + \sum_{n=0}^{\frac{N}{2}-1} s_{n+\frac{N}{2}} \mathrm{e}^{-2\pi\mathrm{i}\frac{(n+\frac{N}{2})m}{N}}
\end{aligned}
$$

Now we will discuss two cases. $m = 2l$ (even) and $m = 2l + 1$ (odd) for $l = 0, \dots, \frac{N}{2} - 1$.

Case $m = 2l$:

$$
\begin{aligned}
S_{2l} &= \sum_{n=0}^{\frac{N}{2}-1} s_n \mathrm{e}^{-2\pi\mathrm{i}\frac{n2l}{N}} + \sum_{n=0}^{\frac{N}{2}-1} s_{n+\frac{N}{2}} \mathrm{e}^{-2\pi\mathrm{i}\frac{(n+\frac{N}{2})2l}{N}} \\
&= \sum_{n=0}^{\frac{N}{2}-1} s_n \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}} + \sum_{n=0}^{\frac{N}{2}-1} s_{n+\frac{N}{2}} \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}} \underbrace{\mathrm{e}^{-2\pi\mathrm{i}l}}_{1} \\
&= \underbrace{\sum_{n=0}^{\frac{N}{2}-1} (s_n + s_{n+\frac{N}{2}}) \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}}}_{\text{DFT length } \frac{N}{2}}
\end{aligned}
$$

Case $m = 2l + 1$:

$$
\begin{aligned}
S_{2l+1} &= \sum_{n=0}^{\frac{N}{2}-1} s_n \mathrm{e}^{-2\pi\mathrm{i}\frac{n(2l+1)}{N}} + \sum_{n=0}^{\frac{N}{2}-1} s_{n+\frac{N}{2}} \mathrm{e}^{-2\pi\mathrm{i}\frac{(n+\frac{N}{2})(2l+1)}{N}} \\
&= \sum_{n=0}^{\frac{N}{2}-1} s_n \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}} \mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}} + \sum_{n=0}^{\frac{N}{2}-1} s_{n+\frac{N}{2}} \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}} \underbrace{\mathrm{e}^{-2\pi\mathrm{i}l}}_{1} \mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}} \underbrace{\mathrm{e}^{-2\pi\mathrm{i}\frac{1}{2}}}_{-1} \\
&= \sum_{n=0}^{\frac{N}{2}-1} s_n \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}} \mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}} - \sum_{n=0}^{\frac{N}{2}-1} s_{n+\frac{N}{2}} \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}} \mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}} \\
&= \underbrace{\sum_{n=0}^{\frac{N}{2}-1} \mathrm{e}^{-2\pi\mathrm{i}\frac{n}{N}} \left(s_n - s_{n+\frac{N}{2}}\right) \mathrm{e}^{-2\pi\mathrm{i}\frac{nl}{\frac{N}{2}}}}_{\text{DFT length } \frac{N}{2}}
\end{aligned}
$$

Fast Fourier Transformation

Instead of one length N DFT we can apply two length $\frac{N}{2}$ DFTs.

The transformation can be applied again and in the second step we need to apply four length $\frac{N}{4}$ DFTs.

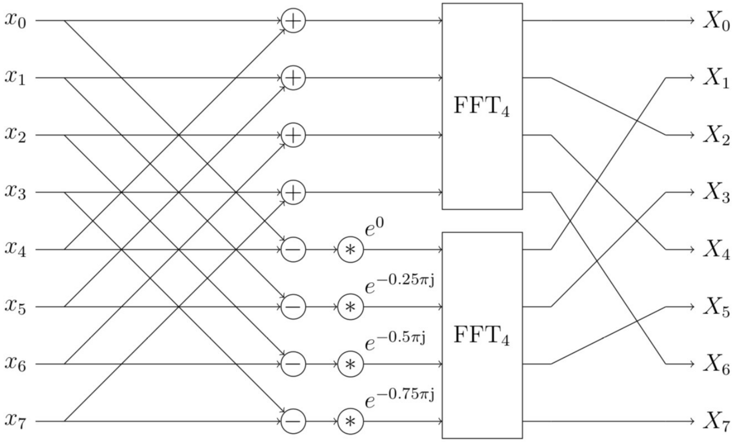
After $r = \log(N)$ steps we need to apply N DFTs of length 1.

In each step the algorithm needs $\frac{N}{2}$ additions, $\frac{N}{2}$ subtractions, and $\frac{N}{2}$ multiplications. The algorithmic complexity it thus $\mathcal{O}(N)$ arithmetic operations.

Since the DFT requires $\log(N)$ steps, the overal time complexity is $\mathcal{O}(N \log(N))$ compared to $\mathcal{O}(N^2)$ of an ordinary DFT.

Fast Fourier Transformation

The FFT is usually implemented inplace and can be visualized as follows



Fast Fourier Transformation

- We have derived the FFT for $N = 2^r$
- The same can be done for other basis 3,5,7,11,13, ...
- By prime factorization one can apply the FFT to general N
- Alternatively, one can pad the vector with zeros to the next power of 2
- The most popular FFT library is the FFTW (Fastest Fourier Transform in the West). It is used in the Julia package FFTW.jl